Computational Methods and Special Functions

by

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ABSTRACT OF THESIS (Regulation 6.9)

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Title of Thesis *Computational Methods and Special Functions*

The material submitted in this thesis, which includes the book *"Sequence Transformations and Their Applications"* as well as published papers, deals with the computation of the special functions of mathematics and physics. Among the various techniques discussed are the Miller algorithm and its extension, summability methods, rational approximations, series expansions, and the inversion of integral transforms.

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10. "On the Computation of Tricomi's Ψ-function," *Computing 13* (1974) 195-203.

11. "Toeplitz Arrays, Linear Sequence Transformations and Orthogonal Polynomials," *Num. Math. 23* (1974) 1-17.

12. "Equiconvergence Theorems for Series Whose Terms Satisfy a Difference Equation," *SIAM J. Math. Anal. 5* (1974) 509-516.

13. "Acceleration Methods," *Encyclopedia of Computer Science and Technology,* Marcel Dekker, N.Y. (1974).

14. "Predictor-Corrector Formulas Based on Rational Interpolants," *Int. J. Comp. Math. Sci. Appl. 1* (1975) 3-12.

15. "Some Transformations of Monotone Sequences," *Math. Comp. 26* (1972) 251-254.

16. "The Summation of Series Whose Terms Have Asymptotic Representations," *Jour. Approx. Theory 10* (1974) 185-198.

17. "An Algorithm for Generating Sequences Defined by Nonhomogeneous Difference Equations," *Rend.Circ. Mat. Palermo 18* (1969) 251-275.

18. "Forward Computation in Second Order Difference Equations," *Jour. Appl. Anal. 1* (1972) 325-329.

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29. "Polynomial Expansions of Bessel Functions and Some Associated Functions," *Math Com. 16* (1962), 446-458.

30. "Expansion of Hypergeometric Functions in Hypergeometric Functions," *Math. Comp. 15* (1961), 390-395.

31. "Polynomial Approximations to Integral Transforms," *Math. Comp. 15* (1961), 174-178.

Description of Thesis

The thesis consists of research into the numerical computation of the special functions of mathematics and physics. We have divided the research effort into several areas. The numbers used refer to the entries in the preceding thesis table of contents.

A. The Miller algorithm

In a number of the included papers we explore the application of the Miller algorithm and its generalizations to the computation of special functions.

The Miller algorithm was devised in 1954 by J.C.P. Miller for the computation of the Bessel functions $I_n(x)$. Since then it has been treated by many authors.

In #20, we survey much of the present field, and discuss in detail the generalization of the algorithm to second order homogeneous difference equations. Much of this material was an extension of research originally submitted in a University of Edinburgh Ph.D. thesis of 1968.

In #18, we further extend these results and show how computational techniques based on non-homogeneous difference equations can be devised for the compuation of such special functions as integrals of Bessel functions.

In #10, we present two methods for the computation of the confluent hypergeometric function $\Psi(a, c; x)$. Both utilize the Miller algorithm. The first, based on a four-term recurrence relation, converges like $O(e^{-an^{2/3}})$, while the second based on a two-term recurrence relation, converges like $O(e^{-an^{1/2}})$.

Much of our research has been devoted to deriving recursion relationships for various special functions in the hope that such formulas will provide the basis for Miller-type algorithms for the computation of these functions. References #25, 24, 23 discuss recurrence relations for generalized hypergeometric functions, and in #9 we derive some differential-difference properties of hypergeometric polynomials.

B. <u>Summability methods</u>

Many special functions may be defined by limiting procedures. Summability methods offer a class of computational techniques for the efficient computation of these limits, and hence the functions defined by them. In addition, these methods offer useful ways of computing many of the important constants of mathematics.

In #16, we show how a Poincaré-type asymptotic form for the n^{th} term of a series induces a similar asymptotic form for the n^{th} partial sum of the series. This expression can then be used to compute the sum of the series.

In #15, we present a method for transforming certain series with monotone terms into series which converge to the same sum but more rapidly.

Entry #13 is a partly expository article, which contains much original research, particularly the idea of summability lozenges and summation arrays based on the classical orthogonal polynomials.

In #11 we discuss in more detail the application of orthogonal polynomials to the construction of lozenge summability algorithms. In #8 a method is discussed for improving the efficiency of some already existing Toeplitz summation procedures and in #6 we show how summability methods may be defined which produce optimal acceleration of convergence of certain sequences arising in Laplace transform theory.

The book included in the thesis is part expository and part original research. It presents the first survey in the English language of the field of applied summability theory, particularly recent important non-linear algorithms.

C. Rational approximations

In #7, we show how a method similar to the Lanczos τ-method for the economization of Taylor series expansions may be applied to Fredholm-type integral equations to yield rational approximations for their solutions.

In #14, we show how non-linear analogues of many of the important formulas of numerical analysis may be constructed. The new formulas are based on rational approximations.

A class of rational approximations to the function $\Psi(a, c; x)$ is derived in #21, and its convergence properties are analyzed.

D. Series expansions

The expansion of a function is a series of more easily computable functions is a time-tested method for devising computational algorithms.

In #27, 29, 30, 31, we show how generalized hypergeometric functions can be expanded in simpler functions of the same type. In #26, by considering the more general G-function of Meijer, we show expansions valid over a semi-infinite ray in the complex plane may be obtained for important classes of functions. The expansions, which are given in terms of Jacobi polynomials, may be interpreted as a sort of summability process wherein the Poincaré type asymptotic expansions possessed by the defining function is re-arranged into a convergent series. Many applications of the theory are given.

The concept of basic series, first popularized by Boas and Buck, is exploited in #28 to obtain expansions for functions not of hypergeometric type. As is the case in #26, basic series may be considered a re-arrangement of a pre-existing power series, convergent or not, into a series of appropriately chosen functions.

In #12, it is shown how simple expansions can be used to obtain more complex expansions. The paper contains as examples some of the first known expansions in Pollaczek polynomials.

In #5 we discuss the computation of sums of the form

$$f(x) = e^{-x} \sum_{n=0}^{\infty} \frac{x^n s_n}{n!}$$

for a wide range of values of the positive variable x, where $\{s_n\}$ is a convergent sequence with the prescribed asymptotic behavior

$$s_n \sim s + \lambda^n \left\{ \frac{C_1}{n} + \frac{C_2}{n^2} + \dots \right\}, \quad n \to \infty.$$

We find that the above asymptotic series induces an asymptotic series on f of the form,

$$f(x) \sim s + e^{x(\lambda-1)} \left\{ \frac{\overline{C}_1}{x} + \frac{\overline{C}_2}{x^2} + \dots \right\}, \quad x \to \infty,$$

for appropriate constants $\overline{C}_j$, and where $s = \lim_{n\to\infty} s_n$.

Such sums are important in scattering theory.

E. <u>Computing a function from its moments and the inversion of integral transforms</u>

In #4 we develop an algorithm for computing a function $\phi(t)$ on $[0, 1]$ at its Lebesque points when its moments

$$c_k = \int_0^1 t^k \phi(t)\, dt$$

are known. Unlike techniques based on Fourier series, the rate of convergence (not just the fact of convergence) depends only on the behavior of the function locally. The method is based on the use of so-called delta-shaped sequences, which have been around for a long time but had always been plagued with unresolved computational problems. As an application, we give a class of algorithms for the inversion of Laplace transforms.

F. Series solutions of differential equations

In the next sequence of papers we examine some properties of series expansions for the solution of differential equations.

In #1 we address the intriguing question: if a Fourier Bessel series

$$\Sigma\, b_n J_\nu (\rho_n x) \quad , \qquad \nu > -\frac{1}{2}\, , \;\; J_\nu(\rho_n) = 0$$

converges to $f(x)$ in the interval $[0, 1]$ of orthogonality of the functions $J_\nu(\rho_n x)$ and converges also at points outside this interval, what does it converge to there? More specifically, if the series converges to 0 in $[1 - \delta, 1]$, does it also converge to 0 in $[1, 1 + \delta]$? Numerical evidence obtained by physicists seemed to indicate the answer was yes. In this paper we show that this conjecture is correct and, surprisingly, is not a consequence of any special properties of Bessel functions but is characteristic of any series of functions satisfying a second order differential equation of Sturm-Liouville type.

In #2, 3 we explore various asymptotic and convergent series representations for solutions of ill-posed and well-posed problems involving the equation of heat conduction.

In #22, we investigate non-trivial representations of zero by series of Gegenbauer polynomials.

G. Zeros of functions

In #19 we establish a class of methods, similar to the Steffensen iteration procedure, for computing the location of the zeros of functions. Unlike the Newton-Raphson process and its generalizations, the methods do not require the evaluation of higher derivatives of the function, yet, for smooth functions, their convergence properties are comparable.

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REMARKS ON THE REPRESENTATION OF ZERO BY SOLUTIONS OF DIFFERENTIAL EQUATIONS

JET WIMP AND DAVID COLTON[1]

ABSTRACT. Numerical evidence from certain problems arising in optics seems to indicate Fourier-Bessel series which converge to zero in $(1-\delta, 1]$ also converge to zero in $[1, 1+\delta)$, an interval which lies outside the range of orthogonality of the Bessel functions. Here we demonstrate this as a corollary of a result on series of functions which satisfy a general Sturm-Liouville equation.

Sometimes it is necessary to determine the behaviour of Fourier-Bessel series

$$\sum b_n J_\nu(\rho_n x), \quad \nu > -\frac{1}{2}, \quad J_\nu(\rho_n) = 0$$

outside the interval [0, 1], the interval of orthogonality of the functions $J_\nu(\rho_n x)$. For instance in a certain problem concerning circularly symmetric positive filters (see [2]) it was given that the above series converged to zero for $x \in [1-\delta, 1]$. In order to guarantee a sufficient spacing in the ring pattern, it was necessary to show that it converged to zero for $x \in [1, 1+\delta]$. Numerical evidence seemed to support this conclusion, and a rigorous proof of this fact has been communicated to the authors by J. Boersma, who based his analysis on the properties of Fourier-Bessel series. In this note we shall show that this result has nothing to do with any special properties (orthogonality, etc.) of Bessel functions, and is in fact true for any sequence of functions satisfying a second order differential equation of Sturm-Liouville type. This result in turn will lead to a rather surprising result on nontrivial representations of zero by infinite series of solutions to ordinary differential equations. In what follows let $\{c_n\}$ be any sequence of real, nonzero constants and $\{y_n\}$ a sequence of functions satisfying

$$Ly_n = c_n^2 y_n, \quad y_n(0) = 0, \quad y_n'(0) = c_n$$

where $L = -d^2/dx^2 + q(x)$ and $q \in C[-\delta, \delta]$. Note that in the analysis which follows the initial condition

$$y_n'(0) = c_n$$

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merely plays the role of a normalization constant.

THEOREM. *If* $\sum|a_n| < \infty$ *and*

$$0 = \sum a_n y_n(x), \quad x \in [0, \delta] \tag{1}$$

then

$$0 = \sum a_n y_n(x), \quad x \in [-\delta, 0]. \tag{2}$$

PROOF. We use the representation [**1**, Appendix IV],

$$y_n(x) = \sin(c_n x) + \int_0^x K(x, t) \sin(c_n t)\, dt, \quad x \in [-\delta, \delta] \tag{3}$$

where $K(x, t)$ is a continuously differentiable function of x and t that is uniquely determined by the function q and the condition $K(0, 0) = 0$. Since K is continuous and $|a_n y_n(x)| \leqslant M|a_n|$,

$$M = 1 + \sup_x \int_0^x |K(x, t)|\, dt, \quad x \in [-\delta, \delta],$$

the series (1) and (2) converge absolutely and uniformly for $x \in [-\delta, \delta]$. Multiplying (3) by a_n and summing shows that

$$0 = u(x) + \int_0^x K(x, t) u(t)\, dt, \quad x \in [0, \delta], u(x) = \sum a_n \sin(c_n x)\cdot$$

and by the uniqueness of solutions to Volterra integral equations this means

$$u(x) = 0, \quad x \in [0, \delta].$$

Furthermore

$$y_n(-x) = -\sin(c_n x) + \int_0^x K(-x, -t) \sin(c_n t)\, dt, \quad x \in [0, \delta],$$

so

$$\sum a_n y_n(-x) = -u(x) + \int_0^x K(-x, -t) u(t)\, dt = 0, \quad x \in [0, \delta].$$

This concludes the proof of the Theorem.

Returning now to our Fourier-Bessel series we note that $y_n(x) = x^{1/2} J_\nu(\rho_n x)$ satisfies the equation

$$Ly_n = \rho_n^2 y_n$$

with $q(x) = x^{-2}(\nu^2 - \frac{1}{4})$. Hence replacing x by $1 - x$, c_n by ρ_n, and using well-known asymptotic properties of Bessel functions we have

COROLLARY 1. *Let* $\sum n^{-1/2}|b_n| < \infty$,

$$\sum b_n J_\nu(\rho_n x) = 0, \quad x \in [1 - \delta, 1].$$

Then the same holds for $x \in [1, 1 + \delta]$.

From the proof of the Theorem we arrive at the surprising

COROLLARY 2. *Let* $\Sigma|a_n| < \infty$. *Then*

$$0 = \sum a_n y_n(x), \quad x \in [0, \delta] \tag{4}$$

if and only if

$$0 = \sum a_n \sin(c_n x), \quad x \in [0, \delta]. \tag{5}$$

In other words a nontrivial representation of zero on an interval in terms of one set of functions leads to another nontrivial representation of zero with the same coefficients but different functions in the expansion. One's first reaction is to suspect that all the coefficients a_n *must* be zero if $\Sigma|a_n| < \infty$ and (4) or (5) is true. But it is quickly seen that this need not be the case if one considers a function $f \in C^2[0, \pi], f \neq 0$, such that $f(\pi) = 0, f(x) = 0$ for $x \in [0, \delta]$ and expands f in a Fourier sine series. In this case we have that $a_n \neq 0$ for all n, $|a_n| = O(n^{-2})$, and (5) is valid for $c_n = n$. It can also be shown that if $f \in C^k[0, 1], f^{(j)}(1) = 0, 0 \leqslant j \leqslant k - 1$, then the Fourier-Bessel coefficients of f are $O(n^{1-k})$. Thus if $k \geqslant 2$ and we make $f(x) > 0$ on $[0, 1 - \delta]$ and zero on $[1 - \delta, 1]$ then $\Sigma n^{-1/2}|b_n| < \infty$ (see Corollary 1) but clearly b_n will not be zero for all n.

It would be interesting to determine necessary conditions on the sequence $\{c_n\}$ such that $\Sigma|a_n| < \infty$ and (5), or equivalently (4), imply that all the a_n are zero.

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The Construction of Solutions to the Heat Equation Backward in Time *)

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Solutions to the backwards heat equation are approximated by solutions of a pseudo-heat equation. Solutions to this modified equation are constructed by means of a fundamental solution and potential theory, and it is shown that the fundamental solution can be approximated by various expansions in special functions.

I Introduction

In recent years there has been a virtual explosion of interest in improperly posed problems in partial differential equations (c.f. [7] and the references contained therein). This interest has been motivated by the ever increasing number of problems that arise in mathematical physics, but are nevertheless improperly posed in the sense that the solution, if it exists, does not depend continuously on the given initial or boundary data. A classical example of such a problem is to determine the temperature of a solid for times t such that $0 < t < t_0$ from a knowledge of its temperature at a fixed time $t_0 > 0$. If we assume that the temperature on the boundary of the solid D is held constant (in particular at zero degrees Centigrade) this problem can be mathematically formulated as follows: Find the temperature $u(x,t)$, $x \in \mathbf{R}^n$, satisfying

$$\text{(1.1a)} \quad \Delta_n u = u_t \qquad \text{in } D x(0,t_0)$$

$$\text{(1.1b)} \quad u = 0 \qquad \text{on } \partial D x(0,t_0)$$

$$\text{(1.1c)} \quad u(x,t_0) = \theta(x) \qquad \text{in } D$$

where $\theta(x)$ is a prescribed function. As is well known ([7]) for a given $\theta(x)$ in general no solution exists, and if a solution does exist it does not depend continuously on the data given at time $t = t_0$ in any reasonable norm. In recent years a variety of methods have been developed to overcome these problems (c.f. [7]), one of the more attractive of these being the method of quasi-reversibility as initially

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*) Dedicated to the memory of our "Doctor-Father" Professor Arthur Erdélyi.

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developed by Lattes and Lions ([5]). In one variation of this approach the initial-boundary value problem (1.1a)–(1.1c) is replaced by the problem

(1.2a) $\varepsilon \triangle_n u_t - u_t + \triangle_n u = 0$ in $D x(0,t_0)$

(1.2b) $u = 0$ on $\partial D x(0,t_0)$

(1.2c) $u(x,t_0) = \theta(x)$ in D

where ε is a small positive parameter. The initial-boundary value problem (1.2a)–(1.2c) is well posed, and it is hoped that for ε sufficiently small the solution of (1.2a)–(1.2c) approximates in some sense the solution of (1.1a)–(1.1c) (if it exists!). To this end Ewing has recently proved the following theorem ([4]):

Theorem *Let* $u(x,t)$ *be a solution of* (1.1a), (1.1b) *such that* $\|u(x,t_0) - \theta(x)\|_{L^2} < \delta$, $\|u(x,0)\|_{L^2} < M$, *where* δ, M *are positive constants, and let* $v(x,t)$ *be the solution of* (1.2a)–(1.2c) *for* $\varepsilon = [\log M/\delta]^{-1}$. *Then for every* $t > 0$,

$$\|u - v\|_{L^2} = O([-\log(\delta/M)]^{-1}).$$

Since in practice the constant M can be determined from a priori physical knowledge, this result gives a convenient approach for approximating the solution to (1.1a)–(1.1c), provided we can solve (1.2a)–(1.2c). For the case $n = 1$ numerical results using this method were provided in [4], where the solution to (1.2a)–(1.2c) was approximated by means of a partial eigenfunction expansion. However in higher dimensions this approach is unsuitable except for very simple geometries due to the fact that the eigenfunctions and eigenvalues are in general not available, and the labour needed to compute these and the corresponding Fourier coefficients becomes formidable. Furthermore if the domain D is changed all calculations have to be repeated again from the beginning. In this paper we shall provide an alternate method for solving (1.2a)–(1.2c) which is based on the use of the fundamental solution for (1.2a) introduced in [2]. This leads to an integral equation of Fredholm-Volterra type for the solution of (1.2a)–(1.2c) and we shall present this analysis in Section II of this paper. In order to construct an approximate solution to this integral equation it is necessary to have a method for accurately approximating the kernel of the integral equation, and difficulties arise here due to the fact that ε is a small parameter. This problem will be studied in Section III. Although we shall restrict our analysis to the case of major practical interest $n = 3$, our analysis can be suitably modified to apply in an arbitrary number of dimensions.

II Polynomial Solutions, Fundamental Solutions, and Pseudo-Heat Potentials

From [2] we can define a fundamental solution of the pseudoheat equation (1.2a) for $n = 3$ by

$$(2.1)\quad \Gamma(R,t) = -\frac{1}{\pi i R} \oint_{\left|\omega - \frac{1}{\sqrt{\varepsilon}}\right| = \delta} \exp\left(-\omega R + \frac{\omega^2 t}{1 - \varepsilon\omega^2}\right) d\omega$$

where $R = |x - \xi|$ for $x, \xi \in R^3$ and the path of integration is a circle of radius δ about the point $1/\sqrt{\varepsilon}$. For future reference we note that

$$\Gamma(R,0) = 0$$

$$\Gamma_t(R,0) = \frac{1}{\varepsilon^{3/2} R} \exp\left(-\frac{R}{\sqrt{\varepsilon}}\right). \tag{2.2}$$

In order to use (2.1) to reformulate (1.2a)–(1.2c) as an integral equation we first need to convert this initial-boundary value problem to one which has zero initial data. In view of Ewing's Theorem we lose little generality in assuming $\theta(x)$ to be approximated by a polynomial $\theta_M(x)$, and in this case we see that

$$u_0(x,t) = \int_{\prod_i^3 C_i} \exp\left(\omega \cdot x + \frac{\omega^2(t - t_0)}{1 - \varepsilon\omega^2}\right) \Phi_M(\omega)\,d\omega_1\,d\omega_2\,d\omega_3 \tag{2.3}$$

is a solution of (1.2a) for $n = 3$ such that $u_0(x,t_0) = \theta_M(x)$ provided we choose $\Phi_M(\omega)$ to be the Borel transform of $\theta_M(x)$ (c.f. [1]) and C_i to be the circle $|\omega_i| < 1/\sqrt{3\varepsilon}$. We note that in (2.3), $\omega = (\omega_1, \omega_2, \omega_3), \omega^2 = \omega \cdot \omega$, and recall that if

$$\theta_m(x) = \sum_{|m| \leqslant M} a_m \frac{x^m}{m!} \tag{2.4}$$

then

$$\Phi_m(\omega) = \sum_{|m| \leqslant M} a_m \omega^{-m-1} \tag{2.5}$$

where $m = (m_1, m_2, m_3)$, $x^m = x_1^{m_1} x_2^{m_2} x_3^{m_3}$, $m! = m_1! m_2! m_3!$, $a_m = a_{m_1 m_2 m_3}$, and $|m| = m_1 + m_2 + m_3$. In particular we see that (2.3) is a polynomial solution of (1.2a) for $n = 3$. Hence by considering $u(x,t) - u_0(x,t)$ and replacing t by $t_0 - t$ we can transform (1.2a)–(1.2c) for $n = 3$ into the initial-boundary value problem

$$\varepsilon \Delta_3 u_t - u_t - \Delta_3 u = 0 \quad \text{in } D \times (0,t_0) \tag{2.6a}$$

$$u = f(x,t) \quad \text{on } \partial D \times (0,t_0) \tag{2.6b}$$

$$u(x,0) = 0 \quad \text{in } D \tag{2.6c}$$

where $f(x,t) = -u_0(x,t_0 - t)$. We now assume that D is a bounded, simply connected domain in $\mathbf{R}^3$ with Liapounov boundary ∂D and let ν denote the unit normal on ∂D pointing into D. Then for any continuous density $\varrho(\xi,\tau)$, the pseudo-heat potential

$$u(x,t) = \frac{1}{2\pi} \int_0^t \int_{\partial D} \varrho(\xi,\tau) \frac{\partial^2}{\partial\nu\,\partial\tau} \Gamma(R,\tau - t)\,ds\,d\tau \tag{2.7}$$

is a solution of (2.6a) satisfying the initial data (2.6c). (In order to take care of a technical point which may arise in the mind of the reader, we note that we can add an arbitrary harmonic function $h(x)$ to (2.7) and still have a solution of (2.6a). Hence even though $u_0(x,t)$ does not vanish on ∂D, we can choose $h(x) = -u_0(x,t_0)$ on ∂D and thus assume that $u(x,0) = f(x,0)$ on ∂D. Since by assumption $u_0(x,t_0)$

is small on ∂D, we have by the maximum principle for harmonic functions that $h(\boldsymbol{x})$ is small in D, and hence neglecting $h(\boldsymbol{x})$ in (2.7) is permissible from the point of view of approximation. We note further that the analysis which follows shows that $\varrho(\boldsymbol{\xi},\tau)$ is independent of $h(\boldsymbol{x})$.) From (2.2) and the discontinuity properties of metaharmonic potentials we can now easily conclude that the boundary condition (2.6 b) will be satisfied provided $\varrho(\boldsymbol{\xi},\tau)$ is a solution of the integral equation

$$
\begin{aligned}
\varepsilon^{3/2}\frac{\partial f(\boldsymbol{x},t)}{\partial t} &= \varrho(\boldsymbol{x},t) + \frac{1}{2\pi}\int_{\partial D}\varrho(\boldsymbol{\xi},t)\frac{\partial}{\partial \nu}\left[\frac{1}{R}\exp\left(-\frac{R}{\sqrt{\varepsilon}}\right)\right]\mathrm{d}s \\
&\quad + \varepsilon^{3/2}\int_0^t \varrho(\boldsymbol{x},\tau)K_{t\tau}(0,\tau-t)\,\mathrm{d}\tau \\
&\quad + \frac{\varepsilon^{3/2}}{2\pi}\int_0^t\int_{\partial D}\varrho(\boldsymbol{\xi},\tau)\frac{\partial}{\partial \nu}\left(\frac{1}{R}\right) \\
&\quad \cdot\left[K_{t\tau}(R,\tau-t) - RK_{Rt\tau}(R,\tau-t)\right]\mathrm{d}s\,\mathrm{d}\tau \\
&= (\boldsymbol{I} + \boldsymbol{T} + \boldsymbol{L}_1 + \boldsymbol{L}_2)\varrho\,,
\end{aligned}
\tag{2.8}
$$

where $K(R,t) = R\Gamma(R,t)$ and

$$
\frac{\partial^3\Gamma(R,\tau-t)}{\partial\nu\,\partial t\,\partial\tau} = \frac{\partial}{\partial\nu}\left(\frac{1}{R}\right)\left[K_{t\tau}(R,\tau-t) - RK_{Rt\tau}(R,\tau-t)\right]. \tag{2.9}
$$

Since from potential theory for metaharmonic functions we know that $(\boldsymbol{I}+\boldsymbol{T})^{-1}$ exists and since $\boldsymbol{L}_1$ and $\boldsymbol{L}_2$ are Volterra operators we can conclude (c.f. [2]) that $(\boldsymbol{I}+\boldsymbol{T}+\boldsymbol{L}_1+\boldsymbol{L}_2)^{-1}$ exists, and hence there exists a unique solution to the integral equation (2.8). An approximate solution to (2.8) can now be found through the use of finite element approximations (c.f. [8]) provided accurate approximations to the function

$$
K(x,t) = -\frac{1}{\pi \mathrm{i}}\oint_{\left|\omega-\frac{1}{\sqrt{\varepsilon}}\right|=\delta}\exp\left[-\omega x + \frac{\omega^2 t}{1-\varepsilon\omega^2}\right]\mathrm{d}\omega \tag{2.10}
$$

and its derivatives can be found for $t \leqslant 0$, $x \geqslant 0$. This will be the subject of the next section of this paper.

III Approximation of $K(x,t)$

As seen in the previous section, in order to approximate the solufion of the integral equation (2.8) it is necessary to approximate the function $K(x,t)$ as defined by (2.10) for $t \leqslant 0$, $x \geqslant 0$. By making the change of variables

$$
X = \frac{x}{\sqrt{\varepsilon}} \qquad T = \frac{t}{\varepsilon} \qquad \omega = \frac{\mu}{\sqrt{\varepsilon}} \tag{3.1}
$$

we can rewrite (2.10) in the form

$$
\tilde{K}(X,T) = -\frac{\mathrm{e}^{-T}}{\pi\mathrm{i}\sqrt{\varepsilon}}\oint_{|\mu-1|=\delta}\exp\left[-\mu X + \frac{T}{1-\mu^2}\right]\mathrm{d}\mu\,. \tag{3.2}
$$

The problem of approximating (2.10) (or (3.2)) is basically a problem in singular perturbation theory and leads to the appearance of "boundary layers" at $t = 0$ (of thickness ε) and at $x = 0$ (of thickness $\sqrt{\varepsilon}$). Furthermore the approximation problem for x fixed and t variable is complicated by the singular behaviour of $K(x,t)$ for $-t \gg \varepsilon$, reflecting the improperly posed behaviour of (2.6a)–(2.6c) for $\varepsilon = 0$. Hence we have chosen to obtain approximations to (3.2) for T fixed and X variable. Our approach will be to obtain an "inner" expansion of (3.2) valid for $X = 0(1)$ (i.e. $x = 0(\sqrt{\varepsilon})$) and an "outer" expansion valid for X large. These expansions lead to the problem of approximating sums of the form

$$f_N(X,T) = \sum_{k=0}^{N} a_k P_k(X) \tag{3.3}$$

where the $P_k(X)$ are either Bessel functions or modified Bessel functions and the a_k are given constants depending on X and T in a simple manner. Since the functions $P_k(X)$ satisfy a three term recursion relation, such sums are readily evaluated (even for large values of N) by using the Clenshaw-Luke method of backward recursion (c.f. [6], Section 11.8).

We first consider the case when $X = 0(1)$. In this case we have

$$\begin{aligned}\tilde{K}(X,T) &= -\frac{\mathrm{e}^{-T-X}}{\pi\mathrm{i}\sqrt{\varepsilon}} \oint_{|\mu-1|=\delta} \exp\left[(1-\mu)X + \frac{T}{1-\mu^2}\right]\mathrm{d}\mu \\ &= \frac{\mathrm{e}^{-T-X}}{\sqrt{\varepsilon}} \sum_{n=0}^{\infty} a_n(T)\frac{X^n}{n!}\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}a_n(T) &= -\frac{1}{\pi\mathrm{i}} \oint_{|\mu-1|=\delta} (1-\mu)^n \exp\left(\frac{T}{1-\mu^2}\right)\mathrm{d}\mu \\ &= -\frac{1}{\pi\mathrm{i}} \sum_{m=0}^{\infty} \frac{T^m}{m!} \oint_{|\mu-1|=\delta} \frac{(1-\mu)^n}{(1-\mu^2)^m}\mathrm{d}\mu \\ &= \sum_{m=n+1}^{\infty} \frac{T^m(2m-n-2)!}{m!(m-1)!(m-n-1)!\,2^{2m-n-2}} \\ &= \frac{T^{n+1}}{2^n\Gamma(n+2)}\,{}_2F_2\left(\begin{matrix}\frac{n}{2}+\frac{1}{2}, \frac{n}{2}+1\\ n+1, n+2\end{matrix}\middle| T\right)\end{aligned} \tag{3.5}$$

where ${}_2F_2$ denotes a generalized hypergeometric function. In order to conveniently apply the Clenshaw-Luke method it is necessary to represent the coefficients $a_n(T)$ in terms of simpler functions. To this end, using the integral representation of the Beta function and the series expansion of the modified Bessel function $I_{m+\frac{1}{2}}(z)$, we have for $m = 0,1,2,\ldots,$

$$\begin{aligned}a_{2m}(T) &= \frac{T^{m+\frac{1}{2}}}{\Gamma(m+\frac{1}{2})} \int_0^1 (1-\xi)^{m-\frac{1}{2}} \exp\left(\frac{\xi T}{2}\right) I_{m+\frac{1}{2}}\left(\frac{\xi T}{2}\right)\frac{\mathrm{d}\xi}{\xi} \\ a_{2m+1}(T) &= \frac{T^{m+\frac{1}{2}}}{\Gamma(m+\frac{3}{2})} \int_0^1 (1-\xi)^{m+\frac{1}{2}} \exp\left(\frac{\xi T}{2}\right) I_{m+\frac{1}{2}}\left(\frac{\xi T}{2}\right)\mathrm{d}\xi\,.\end{aligned} \tag{3.6}$$

This implies from (3.4) that

$$\tilde{K}(X,T) = \sqrt{\frac{T}{\varepsilon}}\, e^{-T-X} \int_0^1 \sum_{m=0}^{\infty} g_m(\xi,T,X) I_{m+\frac{1}{2}}\left(\frac{\xi T}{2}\right) d\xi \tag{3.7}$$

where

$$g_m(\xi,T,X) = \frac{[4TX^2(1-\xi)]^m \exp\left(\frac{\xi T}{2}\right) m!}{\sqrt{\pi}(2m!)^2 \sqrt{1-\xi}} \left(\frac{\sqrt{1-\xi}}{\xi} + \frac{2TX}{(2m+1)^2}\right) \tag{3.8}$$

In order to utilize (3.7) we must know where to truncate the series under the integral sign and still retain an accurate approximation to $\tilde{K}(X,T)$. The truncated kernel can then be summed by the Clenshaw-Luke method and the integral in (3.7) evaluated by numerical integration. The truncation error can be obtained most easily from (3.5) and the fact that

$$\left| {}_2F_2\left(\begin{matrix} \frac{n}{2}+\frac{1}{2}, \frac{n}{2}+1 \\ n+1, n+2 \end{matrix}\middle|\, T\right)\right| \leqslant e^T. \tag{3.9}$$

This implies that the series in (3.4) is majorized by

$$T e^T \sum_{n=0}^{\infty} \frac{(TX)^n}{2^n (n+1)!\, n!}, \tag{3.10}$$

and hence one can now easily deduce the appropriate truncation point for the series in (3.7).

We now consider the case when X is large and the series in (3.7) can no longer be conveniently approximated by truncation. In this case we have from (3.2) that

$$\begin{aligned} \tilde{K}(X,T) &= -\frac{e^{-T-X}}{\pi i \sqrt{\varepsilon}} \oint_{|\mu-1|=\delta} \exp\left[(1-\mu)X + \frac{T}{2(1-\mu)} + \frac{T}{2(1+\mu)}\right] d\mu \\ &= \frac{e^{-T-X}}{\pi i \sqrt{\varepsilon} X} \oint_{|\mu-1|=\delta} \exp\left[\mu + \frac{TX}{2\mu} + \frac{T}{4-\frac{2\mu}{X}}\right] d\mu. \end{aligned} \tag{3.11}$$

From the generating function for Laguerre polynomials ([3]) we have

$$\begin{aligned} &\exp\left(\frac{T}{4-\frac{2\mu}{X}}\right) = e^{T/4}\left[1 + \frac{T}{4}\sum_{n=1}^{\infty} \frac{1}{n} L_{n-1}^{(1)}\left(-\frac{T}{4}\right)\left(\frac{\mu}{2X}\right)^n\right]; \\ &X > \frac{|\mu|}{2} \end{aligned} \tag{3.12}$$

where $L_n^{(1)}(z)$ denotes Laguerre's polynomial. Hence from (3.11) and (3.12) we have

$$\tilde{K}(X,T) = \frac{e^{-T-X}}{\sqrt{\varepsilon X}} \sum_{n=0}^{\infty} a_n(X,T) X^{-n}; \qquad X > 0 \tag{3.13}$$

where for $n \geqslant 0$

$$(3.14)\quad a_n(X,T) = \frac{T e^{T/4} L_{n-1}^{(1)}\left(-\frac{T}{4}\right)}{n\pi i 2^{n+2}} \oint_{|\mu-1|=\delta} \exp\left[\mu + \frac{TX}{2\mu}\right] \mu^n \, d\mu$$

with the definition that $\frac{1}{n} L_{n-1}^{(1)}(-\frac{T}{4}) = 1$ for $n = 0$. Since ([3], p. 7)

$$(3.15)\quad \exp\left(\mu + \frac{TX}{2\mu}\right) = \sum_{n=-\infty}^{\infty} J_n(\sqrt{-2TX}) \left(\frac{2\mu}{\sqrt{-2TX}}\right)^n$$

where $J_n(z)$ denotes Bessel's function, we have from (3.14) that for $n \geqslant 0$

$$(3.16)\quad \begin{aligned} a_n(X,T) &= \frac{T e^{T/4} L_{n-1}^{(1)}\left(-\frac{T}{4}\right)}{n 2^{n+1}} J_{-n-1}(\sqrt{-2TX}) \left(-\frac{TX}{2}\right)^{(n+1)/2} \\ &= \frac{(-1)^{n+1} T e^{T/4} L_{n-1}^{(1)}\left(-\frac{T}{4}\right)}{n 2^{n+1}} J_{n+1}(\sqrt{-2TX}) \left(-\frac{TX}{2}\right)^{(n+1)/2} \end{aligned}$$

and hence

$$(3.17)\quad \begin{aligned} \tilde{K}(X,T) = {} & \frac{T \exp\left(-\frac{3T}{4} - X\right)}{\sqrt{\varepsilon}} \sum_{n=0}^{\infty} \frac{(-1)^{n+1} L_{n-1}^{(1)}\left(-\frac{T}{4}\right)}{n 2^{n+1}} \\ & \cdot J_{n+1}(\sqrt{-2TX}) \left(\frac{-T}{2X}\right)^{(n+1)/2}. \end{aligned}$$

Although the series (3.17) is convergent for all $X > 0$, it is particularly useful for large values of X, since $J_{n+1}(\sqrt{-2TX})$ is bounded. In fact from the inequalities ([3], p. 14, 207)

$$(3.18)\quad |J_n(\sqrt{-2TX})| \leqslant 1 \qquad \left| L_{n-1}^{(1)}\left(-\frac{T}{4}\right) \right| \leqslant (n+1) e^{-T/8}$$

we have that for $X > -T/8$ the series in (3.17) is dominated by

$$(3.19)\quad e^{-T/8} \sqrt{-\frac{T}{2X}} \left(\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{n 2^{n+1}} \left(-\frac{T}{2X}\right)^{n/2}\right); \qquad X > -\frac{T}{8}.$$

In particular from (3.17) and the asymptotic behaviour of Bessel's function one can easily deduce the asymptotic behaviour of $\tilde{K}(X,T)$ for large X (and fixed T).

In both the expressions (3.6) and (3.17) termwise differentiation is permissable and hence approximations are also available for the derivatives of $\tilde{K}(X,T)$.

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Asymptotic Behaviour of the Fundamental Solution to the Equation of Heat Conduction in Two Temperatures

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I. INTRODUCTION

In [5] it was shown that under certain physically reasonable hypothesis one may consider heat conduction to be governed by one temperature and heat supply by another. It was shown that for an extremely general class of *simple* materials these two temperatures turn out to be equal. However, in the case of a *non-simple* material, in particular one in which the thermodynamic quantities depend on the conductive temperature and its first *two* spatial derivatives, the is no longer the case [2]. In particular for an isotropic material in $\mathbb{R}^3$ this linearized version of the energy equation takes the form

$$c\frac{\partial \phi}{\partial t} = k\Delta_3\phi + ca\frac{\partial}{\partial t}\Delta_3\phi + q(\mathbf{x}, t) \tag{1.1}$$

where $\phi(\mathbf{x}, t)$ is the conductive temperature, $q(\mathbf{x}, t)$ the heat supply, c the specific heat, k the conductivity, and a is the temperature discrepancy relating the conductive temperature to the thermodynamic temperature $\Theta(\mathbf{x}, t)$ by the relation

$$\Theta = \phi - a\Delta_3\phi. \tag{1.2}$$

From the second law of thermodynamics we have the fact that $a > 0$. Equations of the form (1.1) have been the subject of a considerable amount of research in recent years (cf. [1]) and, motivated by the need to develop constructive

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methods for solving initial-boundary value problems, it has been shown that a potential theory can be developed for (1.1) and used to solve the standard initial-boundary value problems associated with this equation (cf. [3, 9]). In order to develop this potential theory various methods have been given for constructing fundamental solutions for (1.1) (cf. [3, 8, 9]) and a particular problem which appears is the need to obtain methods for evaluating such a fundamental solution for small values of a (which is the situation appearing in practical applications). From this point of view it seems to us that the fundamental solution defined in [3] and [4] by means of contour integrals provides the best formulation for obtaining such approximations, and it is the purpose of this paper to obtain complete asymptotic expansions for this fundamental solution to (1.1) for small values of the parameter a. In particular we consider the normalized pseudo-heat operator

$$L[u] = \epsilon \Delta_3 u_t - u_t + \Delta_3 u \tag{1.3}$$

where ϵ is a small parameter and derive asymptotic approximations as ϵ tends to zero for the fundamental solution to this equation defined by [3, 4]

$$\Gamma(R, t) = -\frac{1}{\pi i R} \oint_{|\omega - 1/(\epsilon)^{1/2}| = \delta} \exp\left[-\omega R + \frac{\omega^2 t}{1 - \epsilon\omega^2}\right] d\omega \tag{1.4}$$

where $R = |\mathbf{x} - \boldsymbol{\xi}|$ for $\mathbf{x}, \boldsymbol{\xi} \in \mathbb{R}^3$ and the path of integration is a circle of radius δ traversed counterclockwise about the point $\omega = 1/\epsilon^{1/2}$. Clearly it suffices to consider the function

$$K(R, t) = -\frac{1}{\pi i} \oint_{|\omega - 1/(\epsilon)^{1/2}| = \delta} \exp\left[-\omega R + \frac{\omega^2 t}{1 - \epsilon\omega^2}\right] d\omega \tag{1.5}$$

and by making a change of variables we can rewrite (1.5) in the form

$$K(R, t) = -\frac{\exp[-t/\epsilon]}{(\epsilon)^{1/2}\, \pi i} \oint_{|\mu - 1| = \delta} \exp\left[-\mu \frac{R}{\epsilon^{1/2}} + \frac{t}{\epsilon(1 - \mu^2)}\right] d\mu. \tag{1.6}$$

We note that for well posed (as $\epsilon \to 0$) initial boundary value problems associated with $L[u] = q$ we have $R \geqslant 0$ and $t \geqslant 0$ (cf. [3]).

II. The Asymptotic Behaviour of $K(R, t)$ for $t = 0$ and $R = 0$

We first examine the asymptotic behaviour of $K(R, t)$ for the special cases when $t = 0$ or $R = 0$. The case $t = 0$ is trivial since

$$\begin{aligned} K(R, 0) &= -\frac{1}{(\epsilon)^{1/2}\, \pi i} \oint_{|\mu - 1| = \delta} \exp\left[-\mu \frac{R}{\epsilon^{1/2}}\right] d\mu \\ &= 0. \end{aligned} \tag{2.1}$$

In the applications (cf. [3]) one also needs to evaluate the function $K_t(R, 0)$ and this is also easily computed since (1.6) and (2.1) show that

$$\begin{aligned} K_t(R, 0) &= \frac{-1}{\epsilon^{3/2}\pi i} \oint_{|\mu-1|=\delta} \frac{1}{1-\mu^2} \exp\left[-\mu \frac{R}{\epsilon^{1/2}}\right] d\mu \\ &= \frac{1}{\epsilon^{3/2}} \exp\left[-\frac{R}{\epsilon^{1/2}}\right]. \end{aligned} \tag{2.2}$$

We now turn to the case when $R = 0$. In this case we have

$$\begin{aligned} K(0, t) &= -\frac{\exp[-t/\epsilon]}{(\epsilon)^{1/2}\pi i} \oint_{|\mu-1|=\delta} \exp\left[\frac{t}{\epsilon(1-\mu^2)}\right] d\mu \\ &= -\frac{\exp[-t/\epsilon]}{(\epsilon)^{1/2}\pi i} \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{t}{\epsilon}\right)^m \oint_{|\mu-1|=\delta} (1-\mu^2)^{-m}\, d\mu \cdot \\ &= \frac{\exp[-t/\epsilon]}{\epsilon^{1/2}} \sum_{m=0}^{\infty} \frac{(2m)!}{(m+1)!\, m! m!\, 2^{2m}} \left(\frac{t}{\epsilon}\right)^{m+1} \end{aligned} \tag{2.3}$$

and by Legendre's duplication formula

$$\begin{aligned} K(0, t) &= \frac{\exp[-t/\epsilon]}{(\epsilon\pi)^{1/2}} \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{m!\,\Gamma(m+2)} \left(\frac{t}{\epsilon}\right)^{m+1} \\ &= \frac{t \exp[-t/\epsilon]}{\epsilon^{3/2}}\, \Phi\left(\frac{1}{2}, 2; \frac{t}{\epsilon}\right) \end{aligned} \tag{2.4}$$

where $\Phi(a, c; z)$ denotes the confluent hypergeometric function. From the asymptotic behaviour of $\Phi(1/2, 2; z)$ for large z [7, p. 289] we have, for $t > 0$,

$$K(0, t) = \frac{1}{(\pi t)^{1/2}} \left[\sum_{m=0}^{M} \frac{(\frac{1}{2})_m (\frac{3}{2})_m}{m!} \left(\frac{\epsilon}{t}\right)^m + O\left(\left(\frac{\epsilon}{t}\right)^{M+1}\right)\right] \tag{2.5}$$

where $(\alpha)_m = \Gamma(\alpha + m)/\Gamma(\alpha)$. In the applications one also needs to evaluate the derivatives of $K(0, t)$ and it can be shown that the asymptotic behaviour of the derivative of $K(0, t)$ can be obtained from (2.5) by differentiating termwise.

We conclude this section by deriving an expansion for $K(R, t)$ that is well suited for computational purposes when $R > 0$ and $t = O(\epsilon)$. From (1.6) we have

$$\begin{aligned} K(R, t) &= -\frac{\exp[-t/\epsilon]}{(\epsilon)^{1/2}\pi i} \oint_{|\mu-1|=\delta} \exp\left[-\mu \frac{R}{\epsilon^{1/2}}\right] \sum_{m=0}^{\infty} \frac{1}{(1-\mu^2)^m\, m!} \left(\frac{t}{\epsilon}\right)^m d\mu \\ &= -\frac{\exp[-t/\epsilon]}{(\epsilon)^{1/2}\pi i} \sum_{m=0}^{\infty} \frac{a_m}{(m+1)!} \left(\frac{t}{\epsilon}\right)^{m+1} \end{aligned} \tag{2.6}$$

where

$$a_m = \frac{2\pi i}{m!} \lim_{\mu\to 1} \frac{d^m}{d\mu^m} \left[\frac{(\mu-1)^{m+1}}{(1-\mu^2)^{m+1}} \exp\left(-\mu \frac{R}{\epsilon^{1/2}}\right)\right]$$

$$= -\frac{2\pi i}{m!} \sum_{k=0}^{m} \frac{(m+k)!}{k!(m-k)!\,2^{m+k+1}} \left(\frac{R}{\epsilon^{1/2}}\right)^{m-k} \exp\left(-\frac{R}{\epsilon^{1/2}}\right) \tag{2.7}$$

$$= -\frac{2(\pi)^{1/2} i}{m!} \left(\frac{R}{2(\epsilon)^{1/2}}\right)^{m+1/2} K_{m+1/2}\left(\frac{R}{\epsilon^{1/2}}\right)$$

and $K_{m+1/2}(z)$ denotes the modified Bessel function. Therefore

$$K(R,t) = \frac{2\exp(-t/\epsilon)}{(\epsilon\pi)^{1/2}} \sum_{m=0}^{\infty} \frac{1}{(m+1)!\,m!} \left(\frac{t}{\epsilon}\right)^{m+1} \left(\frac{R}{2(\epsilon)^{1/2}}\right)^{m+1/2} K_{m+1/2}\left(\frac{R}{\epsilon^{1/2}}\right). \tag{2.8}$$

The series (2.8) can be readily approximated for $t = O(\epsilon)$ by truncating the series and, since $(z/2)^{m+1/2} K_{m+1/2}(z)$ satisfies a three term recursion relation, applying the Clenshaw–Luke method of backward recursion (cf. [6, Section 11.8]. In this connection we note that for z real

$$\begin{aligned} \left(\frac{z}{2}\right)^{m+1/2} K_{m+1/2}(z) &= \frac{(\pi)^{1/2} e^{-z}}{2^{2m+1} m!} \int_0^\infty e^{-t}(2z+t)^m t^m \, dt \\ &= \frac{\pi^{1/2}}{2^{2m+1} m!} \int_z^\infty e^{-\mu}(\mu^2 - z^2)^m \, d\mu \\ &\leqslant \frac{\pi^{1/2}}{2^{2m+1} m!} \int_0^\infty e^{-\mu}\mu^{2m} \, d\mu \\ &= \tfrac{1}{2}\Gamma(m+\tfrac{1}{2}) \end{aligned} \tag{2.9}$$

by Legendre's duplication formula.

III. Asymptotic Behaviour of $K(r,t)$ for $R > 0$ and $t > 0$

We now turn to the asymptotic behaviour of $K(R,t)$ for $R > 0$ and $t > 0$. To this end we consider the representation (1.5) and deform the contour onto $C = \{z: z = |r| \exp(\pi i/4 \operatorname{sgn} r), -\infty < r < \infty\}$ (see Fig. 1). We then have, setting $\omega = z/t^{1/2}$, $N = t/\epsilon$, $b = R/t^{1/2}$, that

$$K(R,t) = \frac{1}{\pi i (t)^{1/2}} \int_C \exp[-bz + z^2]\, g_N(z)\, dz \tag{3.1}$$

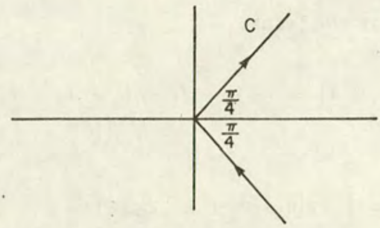

FIGURE 1

where

$$g_N(z) = \exp\left[\frac{z^4}{N - z^2}\right]. \tag{3.2}$$

We now want to examine the behaviour of (3.1) for N tending to infinity. For $|z| < N^{1/2}$ we have

$$\begin{aligned} g_N(z) &= \exp\left[N \sum_{m=2}^{\infty} \frac{z^{2m}}{N^m}\right] \\ &= 1 + \left(\frac{z^4}{N} + \frac{z^6}{N^2} + \cdots\right) + \frac{1}{2}\left(\frac{z^4}{N} + \frac{z^6}{N^2} + \cdots\right)^2 + \cdots \qquad (3.3) \\ &= 1 + c_4 z^4 + c_6 z^6 + \cdots \end{aligned}$$

where the c_{2j}, $j = 2, 3, \ldots$, are easily computable constants. We now write

$$g_N(z) = S_N(z) + T_N(z) \tag{3.4}$$

where

$$\begin{aligned} S_N(z) &= 1 + c_4 z^4 + \cdots + c_{2m} z^{2m}; \qquad m \geqslant 2 \\ T_N(z) &= \sum_{n=m+1}^{\infty} c_{2n} z^{2n}. \end{aligned} \tag{3.5}$$

Since for $|z| < N^{1/2}$

$$|g_N(z)| \leqslant \exp\left[\frac{|z|^4}{N - |z|^2}\right] \tag{3.6}$$

we have by Cauchy's inequality that for $|z| < a < N^{1/2}$

$$\begin{aligned} |T_N(z)| &< \exp\left[\frac{a^4}{N - a^2}\right] \sum_{n=m+1}^{\infty} \left|\frac{z}{a}\right|^{2n} \\ &= \frac{|z/a|^{2m+2}}{1 - |z/a|^2} \exp\left[\frac{a^4}{N - a^2}\right]. \end{aligned} \tag{3.7}$$

We now write $K(R, t)$ in the form

$$K(R, t) = \frac{1}{\pi i (t)^{1/2}} [I_1 + I_2 + I_3 + I_4] \tag{3.8}$$

where

$$\begin{aligned} I_1 &= \int_C \exp[-bz + z^2]\, S_N(z)\, dz \\ I_2 &= \int_{C \cap |z| < A} \exp[-bz + z^2]\, T_N(z)\, dz \\ I_3 &= -\int_{C \cap |z| > A} \exp[-bz + z^2]\, S_N(z)\, dz \\ I_4 &= \int_{C \cap |z| > A} \exp[-bz + z^2]\, g_N(z)\, dz \end{aligned} \tag{3.9}$$

where A is a constant depending on N which will be chosen shortly. First let $A < a$ and consider the integral I_2. Then, since on C

$$|\exp[-bz + z^2]| \leqslant 1, \tag{3.10}$$

we have

$$|I_2| \leqslant 2 \int_0^A \left| \frac{z}{a} \right|^{2m+2} \left[1 - \frac{A^2}{a^2} \right]^{-1} \exp\left[\frac{a^4}{N - a^2} \right] |dz| \tag{3.11}$$

and choosing $a = N^{1/4}$, $A = N^{1/2(2m+3)}$, we have

$$|I_2| = O(N^{-m/2}). \tag{3.12}$$

Now consider I_3. For $N > N_0$ we can find a constant M, independent of N, such that

$$|S_N(z)| \leqslant M |z|^{2m}. \tag{3.13}$$

Hence

$$|I_3| \leqslant 2M \int_A^\infty r^{2m} \exp\left[-\frac{br}{2^{1/2}} \right] dr \tag{3.14}$$

and choosing A as before it is easily seen from the asymptotic behaviour of the incomplete gamma function [7, p. 341] that there exist positive constants ρ and δ, independent of N, such that

$$|I_3| = O(\exp[-\rho N^\delta]). \tag{3.15}$$

Since on C

$$|\exp(z^2)\, g_N(z)| \leqslant 1 \tag{3.16}$$

a similar result is also seen to hold for the integral I_4. Hence from (3.8)–(3.16) we have

$$K(R, t) = \frac{1}{\pi i(t)^{1/2}} \int_C \exp[-bz + z^2]\, S_N(z)\, dz + O(N^{-m/2})$$
$$= a_0 + a_4 c_4 + \cdots + a_{2m} c_{2m} + O(N^{-m/2}) \tag{3.17}$$

where

$$a_{2j} = \frac{1}{\pi i(t)^{1/2}} \int_C \exp[-bz + z^2]\, z^{2j}\, dz \tag{3.18}$$

and the c_{2j} are defined in (3.3). The coefficients a_{2j} may be expressed in terms of Hermite polynomials. To see this we note that the contour C may be deformed onto the imaginary axis and hence from [7, p. 254] we have

$$a_{2j} = \frac{1}{\pi i(t)^{1/2}} \int_{-i\infty}^{i\infty} \exp[-bz + z^2]\, z^{2j}\, dz$$
$$= \frac{(-1)^j}{\pi i(t)^{1/2}} \int_{-\infty}^{\infty} \exp[-ibt - t^2]\, t^{2j}\, dt \tag{3.19}$$
$$= \frac{1}{4^j(\pi t)^{1/2}} \exp\left(-\frac{b^2}{4}\right) H_{2j}\left(\frac{b}{2}\right)$$

where $H_{2j}(z)$ denotes Hermite's polynomial of order $2j$. Hence

$$K(R, t) = \frac{1}{(\pi t)^{1/2}} \exp\left[-\frac{b^2}{4}\right]\left(1 + \frac{c_4 H_4(b/2)}{4^2} + \frac{c_6 H_6(b/2)}{4^3} + \cdots\right.$$
$$\left. + \frac{c_{2m} H_{2m}(b/2)}{4^m} + O(N^{-m/2})\right). \tag{3.20}$$

Now let n be a positive integer and choose m such that $m \geqslant 2n + 2$. Putting all terms with powers of $1/N$ greater than $n + 1$ into the error term shows that the series (3.20) may be rearranged as a valid asymptotic series in $1/N$. Hence returning to our original variables we have our final result that for $R > 0$, $t > 0$,

$$K(R, t) = \frac{1}{(\pi t)^{1/2}} \exp\left(-\frac{R^2}{4t}\right)\left[1 + d_1\left(\frac{\epsilon}{t}\right) + d_2\left(\frac{\epsilon}{t}\right)^2 + \cdots\right.$$
$$\left. + d_n\left(\frac{\epsilon}{t}\right)^n + O\left(\left(\frac{\epsilon}{t}\right)^{n+1}\right)\right] \tag{3.21}$$

where

$$d_1 = \frac{1}{4^2} H_4\left(\frac{R}{2(t)^{1/2}}\right), \qquad d_2 = \frac{1}{2 \cdot 4^4} H_8\left(\frac{R}{2(t)^{1/2}}\right) + \frac{1}{4^3} H_6\left(\frac{R}{2(t)^{1/2}}\right), \text{ etc.}$$

We observe that from

$$H_{2j}(0) = (-1)^j \frac{(2j)!}{j!} \tag{3.22}$$

we have that for $R = 0$, $d_1 = 3/4$, $d_2 = 45/32$, etc. in agreement with (2.5) (Note however that the asymptotic expansion (3.21) has only been shown to be valid for $R > 0$). It can be shown that for $t > 0$ and $R > 0$ the asymptotic series (3.21) can be differentiated termwise.

We note in closing that the asymptotic behaviour of $K(R, t)$ depends critically on the fact that $R \geqslant 0$ and $t \geqslant 0$. If one of these inequalities is violated then the asymptotic behaviour is drastically changed. This is the case for example if the operator (1.3) is used in conjunction with the method of quasi-reversibility to approximate solutions of improperly posed problems (cf. [4]). However, as already mentioned, if potential theoretic methods are used to solve properly posed initial-boundary value problems associated with (1.3) (i.e. stability is required as $\epsilon \to 0$) then we always have R and t non-negative (cf. [3]).

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Computing values of a function on [0, 1] from its moments*

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Synopsis

In this work we present an algorithm for computing an integrable function almost everywhere on (0, 1) when its moments are known. The method is based on the use of certain delta-shaped sequences, and can be adjusted to take advantage of the local smoothness of the function.

As an application, we give an algorithm for the pointwise inversion of the Laplace transform which utilizes the values of the image function at equidistant points.

1. Introduction

In this paper we discuss a class of methods for obtaining pointwise values of a function $\phi \in L[0, 1]$ when its moments

$$\mu_n := \int_0^1 t^n \phi(t) dt, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

are known.

The methods are based on the use of delta-shaped sequences, $\{\delta_n(x, t)\}$, $(x, t) \in (0, 1) \times [0, 1]$, i.e. sequences which, in the limit, mimic the Dirac delta function:

$$\int_0^1 \delta_n(x, t)\phi(t)\, dt = \phi(x) + \mathrm{o}(1), \tag{1.2}$$

at least, when ϕ is sufficiently smooth near x.

The use of such sequences is a very old idea. Widder, in a classical paper, [**1**], used the function

$$\delta_n(x, t) := \left(\frac{n}{x}\right)^{n+1} \frac{e^{-\frac{nt}{x}} t^n}{n!} \quad (x, t) \in (0, \infty) \times [0, \infty) \tag{1.3}$$

to invert the Laplace transform and, in fact, in the same reference used another such sequence to compute ϕ in (1.1).

Historically, the numerical application of delta-shaped sequences suffered from two drawbacks:

1. The rate of convergence is poor. In fact, if one requires (1.2) to hold almost

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everywhere, the most that can be said about the process is that the error is o(1). (Convergence can be improved, however, by taking linear combinations, c.f. the work of C. P. May *et al.* discussed in section 5.)

2. Computing the sequence of iterates (1.2) is usually impractical. Widder's formulas for the computation of ϕ require the evaluation of infinite series.

However for both of these problems, this paper offers some relief, as follows:

1. It is true that convergence of any algorithm based on delta-shaped sequences will be poor, generally speaking, if one insists on computing ϕ where it is not smooth. In reality, however, one seldom wants to find the values at such points. Usually ϕ is piecewise very smooth and one wants its values only in the smoothness intervals. ϕ, for instance, may be a function which satisfies a differential equation on (0, 1) and whose pathology is confined to the endpoints of the interval. An advantage of our algorithm is that it depends on a parameter s which allows one to make use of the local smoothness of the function. In fact, the algorithm has a rather amazing property: it localizes convergence in the sense that as long as ϕ is smooth near x the convergence of the algorithm will depend only very weakly on the behaviour of ϕ away from x. This property we formalize the following definition.

DEFINITION. Let $U_n(\phi, x)$ be an approximation to (an algorithm for computing) $\phi(x)$, $\phi \in M$, $0<x<1$. Define

$$R_n(\phi, x) := U_n(\phi, x) - \phi(x). \tag{1.4}$$

U_n is said to be *entirely local* at x if, for every $\phi_\delta \in M$ having the property

$$\phi_\delta(t) = \phi(t),\ t \in (x-\delta, x+\delta) \subset (0, 1), \tag{1.5}$$

there is a λ_x, $0<\lambda_x<1$, such that

$$R_n(\phi_\delta, x) = R_n(\phi, x) + 0(\lambda_x^n), \tag{1.6}$$

holds as $n \to \infty$.

2. We believe our procedures to be numerically practical. The iterates on the left hand side of (1.2) can be expressed in terms of known values, the moments μ_n; and it can be arranged that the algorithm requires the evaluation of only finite sums.

We have tried to keep this paper uncluttered. Proofs, if they are both tedious and follow an established model, are not given. It is assumed the reader is familiar with the analytical techniques expounded in the basic references, [**2** and **3**].

Notation

$$R_s^{(\alpha,\beta)}(x) := P_s^{(\alpha,\beta)}(2x-1) := \frac{(-1)^s(\beta+1)_s}{s!}\,{}_2F_1\!\left(\begin{matrix}-s,\ s+\alpha+\beta+1\\ \beta+1\end{matrix}\,\middle|\, x\right),\quad x\in[0,1]$$

$$\|f\| := \int_0^1 |f(t)|\,dt$$

$$\delta(f) := \sup_{0\le t\le 1} |f(t)|$$

$$\langle x\rangle := \text{largest integer} \le x,\ x \ge 0$$

$(\alpha)_\nu$:= Pochhammer's symbol $= \alpha(\alpha+1)\ldots(\alpha+\nu-1)$,

$$\nu = 0, 1, 2, \ldots$$

$\{\hat{s}_n\}$:= superior sequence of $\{s_n\}$, $\hat{s}_n = \sup_{k \geqq n} |s_k|$

δ_{ij} := Kronecker delta

Ω := generic positive constant (not necessarily taking the same value whenever it occurs)

Λ := generic constant, $0 < \Lambda < 1$

$\{\Omega_n\}$:= generic bounded sequence

$o, 0, \sim$:= order symbols with respect to n, unless otherwise stated

All special functions are as in [**4**].

2. The Algorithm

Let $f \in L[0, 1]$, $\alpha, \beta > 0$. Define

$$\begin{cases} \mu(\alpha, \beta, f) := \int_0^1 (1-t)^\alpha t^\beta f(t)\, dt, \\ \mu(\alpha, \beta, \phi) := \mu(\alpha, \beta) = \int_0^1 (1-t)^\alpha t^\beta \phi(t)\, dt \\ \mu(\beta) := \mu(0, \beta) = \int_0^1 t^\beta \phi(t)\, dt \\ \mu(n) := \mu_n. \end{cases} \tag{2.1}$$

$\mu(\beta)$ is sometimes called a *moment function.*

We begin our derivation with the following problem: we wish to construct a kernel, $A_s(x, t)$, which is a polynomial of degree s in t such that $\mu(\alpha, \beta, A_s(x, t)\phi(t)) = \phi(x)$ for polynomials $\phi(x)$ of degree $\leqq s$ and such that $\mu(\alpha, \beta, A_s(x, t)\phi(t)) \to \phi(x)$ for a more general class of functions ϕ when α and β tend to infinity in a suitable way.

Define

$$A_s(x, t) = \sum_{j=0}^{s} A_j^s t^j, \quad A_j^s = A_j^s(x). \tag{2.2}$$

Expanding ϕ in a Taylor series about x, we see the condition for polynomials is satisfied if and only if

$$\begin{aligned} \mu(\alpha, \beta, A_s(x, t)(t-x)^\nu) &= \int_0^1 (1-t)^\alpha t^\beta A_s(x, t)(t-x)^\nu\, dt \\ &= \delta_{\nu 0}, \quad \nu = 0, 1, 2, \ldots, s. \end{aligned} \tag{2.3}$$

One's first reaction in trying to solve this problem is to reach for the Jacobi polynomial and make the identification $A_s(x, t) = R_s^{(\alpha,\beta)}(t)$. But this will not do, since (2.3) is violated for $\nu = 0$ and $\nu = s$. What does work is a Christoffel sum of Jacobi polynomials [**4**, v.2, 10.3(10)]:

$$A_s(x, t) = \sum_{k=0}^{s} p_k(x)p_k(t)h_k^{-1}, \tag{2.4}$$

$$= c_s[p_{s+1}(t)p_s(x) - p_{s+1}(x)p_s(t)]/(t-x), \tag{2.5}$$

where

$$p_s(x) = R_s^{(\alpha,\beta)}(x),\ s = 0, 1, \ldots, \tag{2.6}$$

and, in accordance with [**4**, v.2, 10.8],

$$c_s = \frac{(s+1)!\Gamma(\gamma+s+1)}{(\gamma+2s+1)\Gamma(\alpha+s+1)\Gamma(\beta+s+1)},\quad s = 0, 1, 2, \ldots, \tag{2.7}$$

$$\gamma = \alpha + \beta + 1,$$

$$h_k^{-1} := \frac{(2k+y)\Gamma(k+\gamma)k!}{\Gamma(k+\alpha+1)\Gamma(k+\beta+1)},\quad k = 0, 1, 2, \ldots. \tag{2.8}$$

Using (2.5) and the othogonality properties of $p_s(x)$ provide immediate verification of (2.3). Further, (2.4) is the only solution. (Assume another polynomial satisfies (2.3), subtract the two equations, and conclude there is a (non-vanishing) polynomial of degree s orthogonal to all polynomials of degree $\leqq s$, which is impossible).

Note α *and* β *may depend on* x (and, in fact, will). Also

$$A_0^0 = h_0^{-1} = \Gamma(\gamma+1)/\Gamma(\alpha+1)\Gamma(\beta+1). \tag{2.9}$$

Our basic approximation to $\phi(x)$ will be

$$\phi_s(x) := \mu(\alpha, \beta, A_s(x, t)\phi(t)). \tag{2.10}$$

Under certain conditions, as we shall see,

$$\lim_{\alpha,\beta\to\infty} \phi_s(x) = \phi(x). \tag{2.11}$$

(The manner in which the limit is taken is important, and will be dealt with later).

If the moment function $\mu(y)$ is known, $\phi_s(x)$ may be computed from

$$\phi_s(x) = \sum_{j=0}^{s} A_j^s\,\mu(\alpha, \beta+j) = \sum_{j=0}^{s} A_j^s \int_0^1 (1-t)^\alpha t^{\beta+j}\phi(t)\,dt \tag{2.12}$$

since, by dominated convergence,

$$\mu(\alpha, \beta+j) = \sum_{s=0}^{\infty} \frac{(-\alpha)_r}{r!}\mu(\beta+j+r). \tag{2.13}$$

In fact, we will eventually pick β to be an integer. It seems the quantities A_j^s in (2.12) cannot be expressed in a simple closed form. But it is an easy matter to obtain a recursion relationship for them. We multiply (2.5) by $(t-x)$ and equate coefficients of t^j:

$$A_j^s = x^{-1}\left\{A_{j-1}^s + \frac{(-1)^{j+1}(\beta+1)_{s+1}\Gamma(\gamma+s+j)}{\Gamma(\beta+j+1)\Gamma(\alpha+s+1)j!(s+1-j)!}\right.$$
$$\left.\times \sum_{k=0}^{s+1} \frac{(-s-1)_k(s+\gamma)_k(k-j)x^k}{k!(\beta+1)_k}\right\}, \quad j=0,1,\ldots, s, \tag{2.14}$$
$$A_{-1}^s := 0.$$

Putting $j=0$ gives

$$A_0^s = \frac{A_0^0(\gamma+1)_s(\beta+2)_s}{(\alpha+1)_s s!} \sum_{k=0}^{s} \frac{(-s)_k(s+\gamma+1)_k x^k}{k!(\beta+2)_k}. \tag{2.15}$$

These formulae will be of more interest in section 9, where we discuss the inversion of the Laplace transform. For the present moment problem there is a better way of computing ϕ_s, one which avoids the use of A_j^s altogether. We start by using Kummer's transformation on $p_s(t)$.

$$p_s(t) = \frac{(-1)^s(\beta+1)_s}{s!}(1-t)^{-\alpha}{}_2F_1\left(\begin{matrix}\beta+s+1, & -\alpha-s\\ & \beta+1\end{matrix}\,\middle|\, t\right), \tag{2.16}$$

so

$$\phi_s(x) = \int_0^1 t^\beta \sum_{k=0}^{s} \frac{p_k(x)(-1)^k(\beta+1)_k}{h_k k!}$$
$$\times {}_2F_1\left(\begin{matrix}\beta+k+1, & -\alpha-k\\ & \beta+1\end{matrix}\,\middle|\, t\right)\phi(t)\,dt. \tag{2.17}$$

Expanding the ${}_2F_1$ in powers of t, $p_n(x)$ in powers of x, interchanging summation and identifying with (2.1) yields

$$\phi_s(x) = \Gamma(\beta+1)^{-2} \sum_{m=0}^{\infty} \frac{\mu(\beta+m)(-1)^m}{m!(\beta+1)_m}$$
$$\times \sum_{r=0}^{s} \frac{(-1)^r x^r \Gamma(\beta+r+m+1)\Gamma(\gamma+2r)}{r!(\beta+1)_r\Gamma(\alpha+r+1-m)} \tag{2.18}$$
$$\times \sum_{k=0}^{s-r} \frac{(2k+\gamma+2r)(\gamma+2r)_k(\beta+r+m+1)_k}{k!(\alpha+1+r-m)_k}$$

where we assume, for temporary convenience, the legitimacy of the operations and that α is not an integer. The inner sum is now evaluated by a formula from **[5]**:

$$\sum_{k=0}^{l} \frac{(b+2k)(\tau)_k(b+z)_k}{(1-z)_k(b+1-\tau)_k} = \frac{z(b-\tau) + \dfrac{(\tau)_{l+1}(b+z)_{l+1}}{(b+1-\tau)_l(1-z)_l}}{z+\tau}, \tag{2.19}$$

valid as long as everything is defined. Thus

$$\phi_s(x) = A_0^0 \frac{(\gamma+1)_s}{(\alpha+1)_s} \sum_{m=0}^{\infty} \frac{\mu(\beta+m)(\beta+m+1)_{s+1}(-\alpha-s)_m}{m!} K_{m,s}(x), \tag{2.20}$$

$$K_{m,s}(x) := \frac{1}{s!} \sum_{r=0}^{s} (-1)^r \binom{s}{r} \frac{x^r(\gamma+s+1)_r}{(\beta+1)_r(\beta+m+r+1)}. \tag{2.21}$$

For any value of s, the polynomials $K_{m,s}(x)$ are easily written out.

We now wish to find bounds for $K_{m,s}$ and then for $\|\delta_s(x, t)\|$, where

$$\delta_s(x, t) := (1-t)^\alpha t^\beta A_s(x, t). \tag{2.22}$$

We find

$$K_{m,s}(x) = \frac{(-1)^s}{(\beta+1)_s} \int_0^1 t^{\beta+m} R_s^{(\alpha+1,\beta)}(xt)\, dt \tag{2.23}$$

and using the inequality [**4**, v.2, p. 206 (12] yields the bound

$$|K_{m,s}(x)| \leqq \binom{s+p}{s} \Big/ (\beta+1)_s(\beta+m+1), \qquad p = \max(\alpha+1, \beta), \quad x \in [0, 1]. \tag{2.24}$$

Next, using the representation (2.4) in δ_s and majorizing the series gives

$$|\delta_s(x, t)| \leqq \frac{(1-t)^\alpha t^\beta (q+2)_s(\gamma+1)_s A_0^0,}{(\gamma-q)_s s!} \qquad q = \max(\alpha, \beta), \quad t \in [0, 1], \tag{2.25}$$

and integrating gives

$$\|\delta_s(x, t)\| \leqq \frac{(q+2)_s(\gamma+1)_s}{(\gamma-q)_s s!}, \quad x \in [0, 1] \tag{2.26}$$

To explore the properties of ϕ_s we need one final inequality. Note $(1-t)^\alpha t^\beta$ has an absolute maximum at

$$t_0 := \beta/(\alpha+\beta), \quad 0 < t_0 < 1. \tag{2.27}$$

For t_0 and any δ such that

$$0 < t_0 - \delta < t_0 + \delta < 1, \tag{2.28}$$

and $f \in L[0, 1]$,

$$\left| \int_0^{t_0-\delta} (1-t)^\alpha t^\beta A_s(x, t) f(t)\, dt \right| \leqq (t_0-\delta)^{\alpha+\beta} \left[\frac{1}{(t_0-\delta)} - 1 \right]^\alpha \times \frac{(q+2)_s(\gamma+1)_s A_0^0 \|f\|}{(\gamma-q)_s s!}. \tag{2.29}$$

A similar inequality holds for $\int_{t_0+\delta}^1$, $-\delta$ above being replaced by δ.

In what follows $\tau_1 \equiv \tau_1(n, x)$, $\tau_2 \equiv \tau_2(n, x)$ are positive quantities. x is fixed and

τ_1, τ_2 are assumed to be bounded and bounded away from 0 for all n. (We discuss choices of τ_1 and τ_2 in Section 3).

LEMMA 1. *Let* t_0, δ, f *be as in* (2.27)–(2.29), $\alpha = n\tau_1$, $\beta = n\tau_2$. Then

$$\int_0^{t_0-\delta} (1-t)^\alpha t^\beta A_s(x,t) f(t)\, dt = n^{s+\frac{1}{2}} 0(\Lambda^n), \quad n \to \infty. \tag{2.30}$$

The same is true for $\int_{t_0+\delta}^1$.

Proof. Using (2.29), (2.9) and Stirling's formula, we may deduce (2.30) is

$$n^{s+\frac{1}{2}} 0(U(\delta)^n), \quad n \to \infty \tag{2.31}$$

$$U(\delta) := (t_0-\delta)^{\tau_2}[1-(t_0-\delta)]^{\tau_1}/\tau_0^{\tau_2}(1-t_0)^{\tau_1}, \tag{2.32}$$

and as δ increases from 0 to t_0, $U(\delta)$ decreases monotonically from 1 to 0. Thus for δ fixed and

$$0 < \delta < t_0, \tag{2.33}$$

we have

$$0 < U(\delta) < 1. \tag{2.34}$$

We first dispose of the case $s = 0$. This case is of minimal practical but great theoretical interest, see section 3.

THEOREM 1. *Let* $x \in (0, 1)$, $\phi \in L[0, 1]$, $\alpha = n\tau_1$, $\beta = n\tau_2$, $\tau_1 \equiv \tau_1(n, x)$, $\tau_2 \equiv \tau_2(n, x)$, $\lim_{n\to\infty} \tau_2/(\tau_1+\tau_2) = x$.

Let
(*a*) *x be a Lebesgue point of* ϕ, *or, in a neighbourhood of* $t = x$ *let* ϕ *be*
(*b*) *continuous*
(*c*) *of bounded variation*
(*d*) *satisfy a Lipschitz condition of order* ν.
Then respectively:

(*a*), (*b*) $\phi_0(x) = \phi(x) + o(1), \quad n \to \infty$

(*c*) $\phi_0(x) = \dfrac{\phi(x^+) + \phi(x^-)}{2} + o(1), \quad n \to \infty$

(*d*) $\phi_0(x) = \phi(x) + 0(n^{-\nu/2}), \ n \to \infty$

Proof. The proofs are standard. We will give *c*), but comment only briefly on (a) and (d) from (2.27), $t_0 = \tau_2/(\tau_1+\tau_2)$.

To show (a), first assume $t_0 = x$.

By virtue of (2.31), we may write,

$$\phi_0(x) = A_0^0 \int_{x-\delta}^{x+\delta} (1-t)^\alpha t^\beta \phi(t)\, dt + 0(\Lambda^n) \tag{2.35}$$

or

$$\phi_0(x)-\phi(x)=A_0^0\left\{\int_0^\delta \Phi(x,u)\,du+\int_0^\delta \Phi(x,-u)\,du\right\}+0(\Lambda^n), \tag{2.36}$$

$$\Phi(x,u):=[1-(u+x)]^\alpha(u+x)^\beta[\phi(u+x)-\phi(x)]. \tag{2.37}$$

We then write

$$A_0^0\int_0^\delta \Phi(x,u)\,du=A_0^0\int_0^{xn^{-\frac{1}{2}}}\Phi(x,u)\,du+A_0^0\int_{xn^{-\frac{1}{2}}}^\delta \Phi(x,u)\,du \tag{2.38}$$

and similarly for the second integral in (2.36). Proceeding exactly as in [**3**, p. 415] one shows the right hand side of (2.36) is o(1). If, merely, $t_0=x+o(1)$ the appropriate o(1) estimates for the integrals over $[t_0, x]$ or $[x, t_0]$ may be made in (2.35).

(b) now follows from (a).

To show (c), assume $t_0=x$. As in case (a) the more general situation causes no difficulty. The application of a formula in [**6**, p. 37] shows

$$\int_x^1 \delta_0(x,t)\,dt=\frac{1}{2}\int_0^1 \delta_0(x,t)+o(1). \tag{2.39}$$

Assume, as usual, $\phi\uparrow$ near x. Let

$$\begin{aligned}\phi_0(x)&:=I_1(x)+I_2(x),\\ I_2(x)&:=\int_x^1 \delta_0(x,t)\phi(t)\,dt,\\ g(t)&:=\phi(t)-\phi(x^+).\end{aligned} \tag{2.40}$$

Then

$$\begin{aligned}I_2(x)-\frac{\phi(x^+)}{2}&=\int_x^1 \delta_0(x,t)g(t)\,dt+o(1)\\ &=\int_x^{x+\eta}+\int_{x+\eta}^1+o(1)\\ &=g(x^+)\int_x^{x+\xi}\delta_0(x,t)\,dt+g(x+\eta)\int_{x+\xi}^{x+\eta}\delta_0(x,t)\,dt\\ &\quad+o(1),\quad \text{for some } \xi,\quad 0<\xi<\eta,\\ &=g(x+\eta)\int_{x+\xi}^{x+\delta}\delta_0(x,t)\,dt+o(1),\end{aligned} \tag{2.41}$$

by (2.30) and the mean value theorem. Using (2.26) with $s=0$ gives

$$\left|I_2(x)-\frac{\phi(x^+)}{2}\right|\leqq g(x+\eta)+\xi_n,\ \xi_n \text{ a null sequence.} \tag{2.42}$$

Now pick η so the first term is $<\frac{\varepsilon}{2}$, and then n_0 so the second term is $<\frac{\varepsilon}{2}$ for $n>n_0$. This shows $I_2\to\phi(x^+)/2$. A similar argument is used to show $I_1\to\phi(x^-)/2$.

In the proof of (d) we may assume ϕ has the indicated smoothness over the entire interval $[0, 1]$ since, by (2.31), the error will be $0(\Lambda^n)$. Next, it is necessary to estimate the integral $\mu(\alpha, \beta, (t-x)^\nu)$ by Laplace's method. This is straightforward, however, and results in c).

To derive an algorithm for general s, we must make an assumption about the way $\tau_2/(\tau_1+\tau_2)\to x$ as $n\to\infty$. We will require the manner of approach to be algebraic.

LEMMA 2. *Let* $x\in(0, 1)$, $\alpha = n\tau_1$, $\beta = n\tau_2$, *where* τ_1, τ_2, *are as above.* If

$$1-\frac{x(\tau_1+\tau_2)}{\tau_2}=an^{-\theta}(1+o(1)),\quad a\neq 0,\quad \theta>0,\quad n\to\infty, \tag{2.43}$$

then

$$p_s(x):=R_s^{(n\tau_1,n\tau_2)}(x)=C_sn^{\sigma(s)}(1+o(1)),\quad n\to\infty. \tag{2.44}$$

where C_s, $\sigma(s)$ are given in the accompanying Table 1.

In particular if $x=\tau_2/(\tau_1+\tau_2)$, $(\theta=\infty)$,

$$R_s^{[n\tau_2(1/x-1),n\tau_2]}(x)=\zeta_s\frac{\left[\dfrac{(x-1)n\tau_2}{2}\right]^{\langle s/2\rangle}}{\left\langle\dfrac{s}{2}\right\rangle!}(1+o(1)), \tag{2.45}$$

TABLE 1

$$p_s(x)=R_s^{(n\tau_1,n\tau_2)}(x)=C_sn^{\sigma(s)}[1+o(1)],$$

$$1-\frac{x(\tau_1+\tau_2)}{\tau_2}=an^{-\theta}(1+o(1)),\ u_s(x)=\frac{\left[\dfrac{\tau_2(x-1)}{2}\right]^{\langle s/2\rangle}}{\left\langle\dfrac{s}{2}\right\rangle!}$$

e	s	$\sigma(s)$	C_s
$\theta>1$, $x\neq\frac12$	even	$\left\langle\frac{s}{2}\right\rangle$	$u_s(x)$
	odd	$\left\langle\frac{s}{2}\right\rangle$	$u_s(x)(2x-1)(2s+1)/3$
$\theta>1$, $x=\frac12$	even	$\left\langle\frac{s}{2}\right\rangle$	$u_s(\frac12)$
	odd	$\left\langle\frac{s}{2}\right\rangle+1-\theta$	$-\tau_2au_s(\frac12)$
$0\leqq\theta\leqq\frac12$		$s(1-\theta)$	$(-1)^s(\tau_2a)^s/s!$
$\frac12<\theta<1$	even	$\left\langle\frac{s}{2}\right\rangle$	$u_s(x)$
	odd	$\left\langle\frac{s}{2}\right\rangle+1-\theta$	$-\tau_2au_s(x)$
$\theta=1$	even	$\left\langle\frac{s}{2}\right\rangle$	$u_s(x)$
	odd	$\left\langle\frac{s}{2}\right\rangle$	$\left[\frac{(2x-1)(2s+1)}{3}-\tau_2a\right]u_s(x)$

where

$$\zeta_s = \begin{cases} 1, & s \text{ even} \\ \dfrac{(2s+1)(2x-1)}{3}, & s \text{ odd.} \end{cases} \tag{2.46}$$

Proof. First by construction, then by induction. We begin with the recursion relation for $p_s(x)$, which we write in the form

$$\begin{aligned} p_{s+1}(x) = \frac{-n}{(s+1)}\{[\tau_2 a n^{-\theta}(1+o(1)) + (1-2x)(2s+1)n^{-1} \\ + 0(n^{-1-\delta})]p_s(x) + \tau_2(1-x)(1+0(n^{-1}))p_{s-1}(x)\}, \ \delta > 0. \end{aligned} \tag{2.47}$$

We will construct inductively an asymptotic relationship (2.44) by determining C_s, $\sigma(s)$ so that they satisfy

$$\begin{aligned} C_{s+1}n^{\sigma(s+1)} = \frac{-1}{(s+1)}\{&\tau_2 a C_s n^{1-\theta+\sigma(s)}(1+o(1)) \\ &+ (1-2x)(2s+1)C_s n^{\sigma(s)}(1+o(1)) \\ &+ \tau_2(1-x)C_{s-1}n^{1+\sigma(s-1)}(1+o(1))\} \end{aligned}$$

starting with

$$\begin{aligned} p_0 &= 1, \\ p_1 &= -\tau_2 a n^{1-\theta}(1+o(1)) + (2x-1). \end{aligned} \tag{2.49}$$

To show the reader how the computations go, we will do the case $\frac{1}{2} < \theta < 1$. The other cases are similar, or simpler.

We must have

$$\sigma(s+1) = \max(1-\theta+\sigma(s), \quad 1+\sigma(s-1)). \tag{2.50}$$

Since $\sigma(0) = 0$, $\sigma(1) = 1-\theta$, the above formula generates the values for $\sigma(s)$ claimed in line four of the table.

Letting $s = 2m+1 = \text{odd}$ in (2.48) and equating leading terms shows

$$C_{2m+2} = \frac{-\tau_2(1-x)}{2(m+1)}C_{2m}, \tag{2.51}$$

or

$$C_{2m} = \frac{\tau_2^m(x-1)^m}{m!\ 2^m}. \tag{2.52}$$

Next let $s = 2m = \text{even}$. The same process yields

$$C_{2m+1} = \frac{-1}{(2m+1)}[\tau_2 a C_{2m} + \tau_2(1-x)C_{2m-1}]. \tag{2.53}$$

We make the substitution

$$C_{2m+1} = \frac{\tau_2^m(x-1)^m}{m!\ 2^m}g_m, \tag{2.54}$$

so (2.53) becomes

$$(2m+1)g_m - 2mg_{m-1} = -\tau_2 a. \tag{2.55}$$

A particular solution of this difference equation is

$$g_m^* = -\tau_2 a, \tag{2.56}$$

and this, in fact, is the solution required, since the formula

$$C_{2m+1} = -\tau_2 a \frac{\tau_2^m (x-1)^m}{m!\, 2^m},$$

checks for $m = 0$. Thus line four of the table is verified.

Note in general C_s may depend on n, and will, if τ_2 does.

It is interesting to note that if $\tau_1 = \tau_2 = \text{const.}$ and $x = \tau_2/(\tau_1 + \tau_2)$, the method of steepest descents may be used on the integral [**4**, v.1, p. 114]

$$\begin{cases} p_s(x) = \dfrac{(-1)^s \Gamma(\beta+s+1)\Gamma(\alpha+s+1)}{2\pi i \Gamma(\gamma+s) s!} \displaystyle\int_0^{(1^+)} e^{nh(t)} H(t)\, dt, \\ h(t) := (\tau_1 + \tau_2) \ln t - \tau_1 \ln (t-1), \\ H(t) := (1-xt)^s (t-1)^{-s-1} t^s, \end{cases} \tag{2.58}$$

and shows the $o(1)$ term in (2.45) is in this case $0(n^{-1})$.

Now let us assume that $\phi^{(s+1)}$ is continuous on $[0, 1]$. Let

$$g(t) := \frac{\phi(t) - \phi(x)}{t - x}. \tag{2.59}$$

Then $g^{(s)}(t)$ is continuous on $[0, 1]$. We have

$$\phi_s(x) - \phi(x) = c_s \int_0^1 (1-t)^\alpha t^\beta \{p_{s+1}(t) p_s(x) - p_{s+1}(x) p_s(t)\} g(t)\, dt, \tag{2.60}$$

and using [**4**, 10.7] we may integrate s times by parts to get

$$\begin{aligned} |\phi_s(x) - \phi(x)| &= \frac{c_s}{(s+1)!} \left| \int_0^1 (1-t)^{\alpha+s} t^{\beta+s} \right. \\ &\qquad \left. \times \left\{ q(t) p_s(x) + (s+1) p_{s+1}(x) \right\} g^{(s)}(t)\, dt \right| \\ &\leqq \frac{c_s}{(s+1)!} \sup_{0 \leqq t \leqq 1} |g^{(s)}(t)| \int_0^1 (1-t)^{\alpha+s} t^{\beta+s} \\ &\qquad \times \left\{ |q(t)|\, |p_s(x)| + (s+1)\, |p_{s+1}(x)| \right\} dt, \end{aligned} \tag{2.61}$$

where

$$\begin{aligned} q(t) &= (\beta+s+1) - (\gamma+2s+1)t \\ &= -p_1(t) - s(2t-1) \\ &= -p_1(x) - (\gamma+2s+1)(t-x) - s(2t-1) \end{aligned}$$

so, by (2.49):

$$|q(t)| \leqq \Omega n^{1-\theta} + (s+1) + (\gamma+2s+1)\, |t-x|.$$

To estimate the contribution to (2.61) arising from the last term on the right hand side above we compute:

$$\int_0^1 (1-t)^{\alpha+s}t^{\beta+s}(t-x)\,dt \leqq |t_0-x|\int_0^1 (1-t)^{\alpha+s}t^{\beta+s}\,dt$$
$$+\int_0^1 (1-t)^{\alpha+s}t^{\beta+s}|t_0-t|\,dt.$$

The first integral is $B(\alpha+s+1,\ \beta+s+1)\times 0(n^{-\theta})$ and the second when broken up into $\int_0^{t_0}$ and $\int_{t_0}^1$, may be dealt with by Laplace's method, see [**6**, p. 37], and is $B(\alpha+s+1,\ \beta+s+1)\times 0(n^{-\frac{1}{2}})$.

Thus (2.61) becomes

$$|\phi_s(x)-\phi(x)| = \{|p_s(x)|[0(n^{1-\theta})+0(n^{\frac{1}{2}})]+|p_{s+1}(x)|0(1)\}n^{-s-1},$$
$$= 0(n^{\nu(s)}),$$
$$\nu(s) = \max\,(\sigma(s)+1-\theta,\ \sigma(s)+\tfrac{1}{2},\ \sigma(s+1))-s-1.$$

Thus $\nu(s)$ may be computed using the values of $\sigma(s)$ in the table,† and we have:

THEOREM 2. *Let $x\in(0,1)$, $\alpha=n\tau_1$, $\beta=n\tau_2$, where τ_1, τ_2 are positive, bounded and bounded away from zero. Let $\phi\in L[0,1]$ and let $\phi^{(s+1)}$ be continuous in a neighbourhood of x. If*

$$1-\frac{x(\tau_1+\tau_2)}{\tau_2} = an^{-\theta}(1+o(1)),\ a\neq 0,\ \theta\geqq 0,\ n\to\infty. \tag{2.62}$$

then

$$\begin{cases}\phi_s(x)=\phi(x)+0(n^{\nu(s)}),\ n\to\infty,\\ \nu(s)=-(s+1)\min\,(\theta,\tfrac{1}{2}).\end{cases} \tag{2.63}$$

In particular, if $x=\tau_2/(\tau_1+\tau_2)$,

$$\phi_s(x)=\phi(x)+0(n^{-\frac{(s+1)}{2}}), \qquad n\to\infty. \tag{2.64}$$

Proof. Only one remark is needed to conclude the proof. We have assumed ϕ smooth throughout $[0,1]$. This is legitimate, since the error arising from this assumption will be exponentially small, see Lemma 1.

This observation becomes

THEOREM 3. *In the Definition, let $M=L[0,1]$, U_n be any algorithm mentioned in Theorem 1 or 2. Then U_n is entirely local at x.*

3. CHOICES OF α, β AND SOME PROPERTIES OF DETERMINING FUNCTIONS

When β is an integer, ϕ_s may be computed in terms of the moments μ_n. When α is an integer, the series in (2.20) terminates, so the algorithm is finite. One

† If $\phi^{(s+2)}$ is continuous at x, then (2.60) may be integrated $(s+1)$ times by parts and this result strengthened to $\nu(s)=-\left\langle\frac{s}{2}\right\rangle-1$.

effective choice is

$$\alpha = \langle n/x \rangle - n, \quad \beta = n, \quad \tau_1 = \frac{\langle n/x \rangle}{n} - 1, \quad \tau_2 = 1, \tag{3.1}$$

so

$$t_0 = n/\langle n/x \rangle = x + 0(n^{-1}). \tag{3.2}$$

and $\theta = 1$ in (2.62). The choice (2.64):

$$\alpha = \frac{n}{x} - n, \quad \beta = n, \quad \tau_1 = \frac{1}{x} - 1, \quad \tau_2 = 1, \quad t_0 = x, \tag{3.3}$$

is also a good one, and the approximation to $\phi(x)$ may then be conveniently rewritten as

$$\phi_s(x) = \frac{(-1)^{n+1}}{(n+s)!} \sum_{m=n}^{\infty} \frac{\left(-1-s-\dfrac{n}{x}\right)_{m+1}}{(m-n)!} \mu_m L_{m,s}(x), \quad s = 0, 1, 2, \ldots, \tag{3.4}$$

$$\begin{aligned} L_{m,s}(x) &= \frac{(m+1)_{s+1}(n+1)_s}{s!} \sum_{r=0}^{s} \binom{s}{r} \frac{\left(\dfrac{n}{x} + s + 2\right)_s}{(n+1)_r (m+r+1)} \\ &= \frac{(m+2)_s (n+1)_s}{s!} {}_3F_2\left(\left. \begin{matrix} -s, & \dfrac{n}{x} + s + 2, & m+1 \\ & n+1, & m+2 \end{matrix} \right| x \right). \end{aligned} \tag{3.5}$$

We find:

$$L_{m,0}(x) = 1, \tag{3.6}$$

$$L_{m,1}(x) = -3x(m+1) + (m+n+2), \tag{3.7}$$

$$\begin{aligned} L_{m,2}(x) &= 10(m+1)(m+2)x^2 + \frac{(m+1)}{2}(mn - 16m - 6n - 48)x \\ &\quad + \tfrac{1}{2}(2n^2 - mn - m^2 n + 6n + 2m^2 + 10m + 2). \end{aligned} \tag{3.8}$$

Using the theory in [**5**], one may obtain a 4-term recursion relation in s for $L_{m,s}(x)$.

The general term of the series (3.4) is $0(\mu_m m^{n-n/x-1})$ in m, so the series always converges. The use of the bound (2.24) for $K_{m,s}$ will provide a convenient estimate of the error incurred by truncating the series (2.20).

Now let us choose α, β as in (3.1) and put $s = 0$ in (2.20). We have

$$\begin{cases} \phi_0(x) = A_0^0 \sum\limits_{m=0}^{\alpha} \mu_{n+m} \dfrac{(-\alpha)_m}{m!} \\ \qquad\quad = \Omega n_\rho^{\frac{1}{2}}(1 + \Omega_n n^{-1}) \sum\limits_{m=0}^{\alpha} \mu_{n+m} \dfrac{(-\alpha)_m}{m!}, \end{cases} \tag{3.9}$$

where

$$\rho \equiv \rho(x) := x^{-1}(1-x)^{1-x^{-1}}. \tag{3.10}$$

$\rho(x)$ is infinite at $x=0$ and decreases monotonically to 1 as $x \to 1$. The form of (3.9) makes clear that if $\mu_n = 0$, $n > n_o$, then $\phi = 0$ almost everywhere, and so: if μ_n is moment sequence, no finite alteration of μ_n can be a moment sequence. This is in contradistinction to moment sequences having representations as Stieltjes integrals,

$$\mu_n = \int_0^1 t^n \, d\alpha(t), \tag{3.11}$$

see [**7**, p. 163].

(3.9) also shows that no finite subset of $\{\mu_n\}$ is required to know ϕ almost everywhere, so any algorithm for computing ϕ which requires the knowledge of any finite subset of moments is inefficient. For instance, the following method is sometimes used to obtain global approximations to ϕ. Assume ϕ has a uniformly convergent expansion in shifted Legendre polynomials

$$\phi(t) = \sum_{h=0}^{\infty} a_k p_k(t), \tag{3.12}$$

$$p_k(t) = P_k(2t-1). \tag{3.13}$$

Then

$$\phi(t) = \sum_{k=0}^{\infty} \mu_k^* p_k(t), \tag{3.14}$$

where

$$\mu_k^* = \frac{(-1)^k}{(k+\frac{1}{2})} \sum_{j=0}^{k} \frac{(-k)_j (k+1)_j}{j!^2} \mu_j. \tag{3.15}$$

Aside from the stringent smoothness required of ϕ if the series (3.14) is to be of practical value, it is obviously inefficient, in the above sense, at every point of $(0,1)$. Global approximations which do not depend on finite subsets of $\{\mu_n\}$ may be obtained from (2.20)–(2.21) by taking $\beta = n$, $\alpha =$ arbitrary. They will not, in general, be entirely local at any point of $(0,1)$.

Now we return to (3.9)–(3.10). We have

$$|\phi_0(x)| \leqq \Omega \rho^n \hat{\mu}_n 2^\alpha n^{\frac{1}{2}} \tag{3.16}$$

so if x is a Lebesgue point of ϕ, and

$$\left[x^{-1}\left[\frac{(1-x)}{2}\right]^{1-x^{-1}}\right]^n n^{\frac{1}{2}} \hat{\mu}_n = o(1), \tag{3.17}$$

then $\phi(x) = 0$. Compare with Widder's discussion of how changes of sign in ϕ affect $\{\mu_n\}$, [**1**, p. 197]. Also (3.9) provides a cheap proof of the fact that any sequence which (ultimately) alternates in sign cannot be a moment sequence. For, since $(-1)^m(-\alpha)_m$ is positive, this would mean $\phi_0(x)$ is ultimately alternating, which would mean $\phi(x) = 0$, or, $\phi(x) = 0$ almost everywhere, which is not possible. The same can be said if any linear difference operator with constant coefficients acting on $\{\mu_n\}$ produces an ultimately alternating sequence, for

$$\Delta^k \mu_n = \int_0^1 t^n (t-1)^k \phi(t) \, dt. \tag{3.18}$$

Furthermore, since

$$\mu_n/\Lambda^n \to 0 \text{ iff } \hat{\mu}_n/\Lambda^n \to 0, \tag{3.19}$$

(3.17) shows no moment sequence $\not\equiv 0$ is of smaller than exponential order. This is true since the quantity inside the brackets in (3.17) is >1, so $\phi(x)=0$ almost everywhere. Any direct proof of this seems laborious.

On the other hand, a direct examination of the integral for $\phi_0(x)$ using simple order estimates and the fact that the algebraic portion of the integrand has a maximum at t_0 shows that if x is a Lebesgue point of ϕ and

$$\lim_{n\to\infty} \frac{n^{\frac{1}{2}}\mu_n}{x^n} = 0, \tag{3.20}$$

then either $\phi(x)=0$ or ϕ is not of one sign in any neighbourhood of x.

If one works to improve Lemma 1 by taking into account integrals like $\int_0^{t_0 - \delta\eta_n^{-1}}$, $\hat{\eta}_n \to \infty$, this result may be strengthened in the following way: if

$$\lim_{n\to\infty} \frac{\mu_n(lnn)^{\frac{1}{2}}}{x^n} e_n = 0 \tag{3.21}$$

for any positive sequence $\{e_n\}$, $e_n \to \infty$, then either $\phi(x)=0$ or ϕ is not of one sign in any neighbourhood of x.

4. Numerical Example

Let

$$\phi(t) = \sqrt{1-t}. \tag{4.1}$$

Then

$$\mu_n = n!/(\tfrac{3}{2})_{n+1}. \tag{4.2}$$

We pick

$$\alpha = \frac{n}{x} - n,\ \beta = n, \tag{4.3}$$

and apply formulae (3.4), (3.5) with $s=4$ and $s=6$.
In Table 2 is tabulated

$$r_{n,s}(x) = \sqrt{1-x} - \phi_s(x). \tag{4.4}$$

Note the values near 0 are quite good, despite the fact that ϕ is singular at $x=1$. Of course, the accuracy deteriorates as we approach the singularity. Nevertheless it is clear the convergence of the algorithm is strongly local.

The traditional method for computing $\phi(t)$ is to such an expansion in (shifted) Legendre polynomials, $P_n(2t-1)$, (see section 3). In this case, however, the error of such an expansion is $0(n^{-\frac{5}{2}})$ uniformly for $0 \leqq t \leqq 1-\delta$, where n is the number of terms taken, and the accuracy deteriorates to $0(n^{-\frac{3}{2}})$ near 1. The presence of the singularity at 1 drastically affects the convergence of the expansion in every sub-interval of [0, 1].

5. The Inversion of Integral Transforms

In the formula

$$\phi_s(x) := \int_0^1 (1-t)^\alpha t^\beta A_s(x,t)\phi(t)\,dt, \tag{5.1}$$

makes the changes of variable

$$\begin{cases} t := 1-e^{-au}, \quad a \text{ arbitrary}, >0, \\ \qquad \phi(1-e^{-au}) := \psi(u), \\ x := 1-e^{-ay}, \quad y = -a^{-1}\ln(1-x), \\ \Psi(p) := \displaystyle\int_0^\infty e^{-pu}\psi(u)\,du, \quad p \geqq 0. \end{cases} \tag{5.2}$$

Then an approximation to ψ is

$$\psi_s(y) := a\int_0^\infty e^{-(\alpha+1)au}(1-e^{-au})^\beta A_s(1-e^{-ay}, 1-e^{-au})\psi(u)\,du. \tag{5.3}$$

Let

$$H_\beta := \int_0^\infty e^{-(\alpha+1)au}(1-e^{-au})^\beta \psi(u)\,du, \tag{5.4}$$

$$= \sum_{k=0}^\infty \frac{(-\beta)_k}{k!}\Psi(a(k+\alpha+1)), \tag{5.5}$$

Then

$$\psi_s(y) = a\sum_{j=0}^s A_j^s H_{\beta+j}, \tag{5.6}$$

and to apply Theorem 2 we let

$$\alpha = n\tau_1, \quad \beta = n\tau_2, \quad \beta/(\alpha+\beta) = x = 1-e^{-ay}.$$

Note if β is an integer, the series (5.5) terminates and the algorithm is finite. Thus if $\psi^{(s+1)}$ is continuous in a neighbourhood of y,

$$\psi_s(y) = \psi(y) + 0(n^{-(s+1/2)}). \tag{5.7}$$

Here the choices (3.1) produce a convenient finite algorithm. This generalizes a result from **[8]**, $(s=0,1,2)$. In **[9]**, May studies the approximation of $\psi(u)$ by finite linear combinations of Widder-Post operators, see equation (1.3). The order of convergence is comparable with our algorithm. For many other operators – those of exponential type – convergence can be strengthened by taking linear combinations. This idea goes back at least to Butzer's 1953 paper on Bernstein polynomials **[10]**. May's paper has a good bibliography.

A similar result obtains for Mellin transforms; let

$$\psi(p) := \int_0^1 u^{p-1}\psi(u)\,du. \tag{5.8}$$

TABLE 2

$$\phi(x)=\sqrt{1-x},\ \alpha=\frac{n}{x}-n,\ \beta=n$$

$$r_{n,s}(x)=\phi(x)-\phi_s(x)$$

	$s=4$				$s=6$		
$x \backslash n$	10	20	30	$x \backslash n$	10	20	30
0·2	3·5(−7)†	5·5(−8)	1·7(−8)	0·2	−4·5(−9)	−5·7(−10)	−1·9(−10)
0·4	1·0(−5)	1·5(−6)	4·9(−7)	0·4	−1·1(−6)	−9.4(−8)	−2·1(−8)
0·6	−3·7(−5)	−6·9(−6)	−2·4(−6)	0·6	1·2(−5)	1·6(−6)	4·0(−7)
0·8	−9·0(−4)	−3·2(−4)	−1·5(−4)	0·8	5·2(−5)	5·2(−5)	5·2(−5)

† $a(m)=a\times 10^m$

Then the substitution $t=u/(1+u)$ in (5.1) and taking $\phi\left(\frac{u}{1+u}\right)=\psi(u)$ yields the approximation

$$\psi_s(y)=\sum_{j=0}^{s} A_j^s I_{\beta+j}, \tag{5.9}$$

$$I_\beta=\sum_{k=0}^{\infty}\frac{(\alpha+\beta+2)k}{k!}(-1)^k\psi(\beta+k+1), \tag{5.10}$$

and the order of approximation is (5.7) when

$$\alpha=n\tau_1,\quad \beta=n\tau_2,\quad \beta/(\alpha+\beta)=x,\quad y=x/(1-x). \tag{5.11}$$

However no permissible values of α and β yield a finite algorithm and there is no guarantee of convergence of the series (5.10).

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Note

The Computation of Borel-Type Sums Arising in Scattering Theory

1. Introduction

It is often necessary, for instance in scattering theory [1], to calculate sums of the form

$$f(x) = e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} s_n \tag{1}$$

for a wide range of values of the positive variable x, where $\{s_n\}$ is some fixed convergent sequence.

We use the notation

$$s_n \leftrightarrow f(x) \tag{2}$$

to indicate relationship (1) and we call f the *Borel transform* of the sequence $\{s_n\}$.

It is known that if

$$\lim_{n\to\infty} s_n = s \qquad \text{then} \qquad \lim_{x\to\infty} f(x) = s; \tag{3}$$

see Knopp, [2, p. 472]. From this point of view the relationship $s_n \leftrightarrow f$ is a summation process which can be used to compute the (generally unknown) value of the limit of the sequence s_n.

The problem presented by sums such as (1) when they occur in physics is usually the inverse of this: s_n is known (generally it is a correlation function) and the task is to compute the function f.

When x is small, the computational problems are not severe. When x is large, the computation of f from its defining series presents grave overflow–underflow problems, and the task is decidedly nontrivial. In many important cases, a technique for computing f may be obtained by asymptotic analysis.

In what follows we use the notation

$$s = \lim_{n\to\infty} s_n,$$

$$s_n\hat{} = \sup_{k>n} |s_k|,$$

$$r_n = s - s_n, \qquad \text{the remainder sequence,}$$

$$f_N(x) = s - e^{-x} \sum_{n=0}^{N} \frac{x^n r_n}{n!}, \tag{4}$$

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$$R_N(x) = -e^{-x} \sum_{n=N+1}^{\infty} \frac{x^n r_n}{n!}, \qquad \text{the remainder function,}$$

$$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \qquad n = 0, 1, 2,\ldots \text{ (Pochhammer's symbol).}$$

The notation for all special functions in this paper is that of [3].

By linearity of the "$\leftrightarrow$" relationship we have

$$\begin{aligned} f_N(x) + R_N(x) &= s - e^{-x} \sum_{n=0}^{\infty} \frac{x^n r_n}{n!} \\ &= s - e^{-x} \sum_{n=0}^{\infty} \frac{x^n(s - s_n)}{n!} \\ &= s - s + e^{-x} \sum_{n=0}^{\infty} \frac{x^n s_n}{n!} = f(x). \end{aligned} \tag{5}$$

2. Computation of f for x Small

If x is not too large, f_N is a good approximation to f for N suitably large. We have, in fact,

$$\begin{aligned} |f(x) - f_N(x)| = |R_N(x)| &\leqslant e^{-x} \sum_{n=N+1}^{\infty} \frac{x^n |r_n|}{n!} \\ &\leqslant r_N\hat{}\, e^{-x} \sum_{n=0}^{\infty} \frac{x^{N+n+1}}{(n+N+1)!}. \end{aligned} \tag{6}$$

Using the fact that

$$(u+v)! \geqslant u!\, v!, \tag{7}$$

we have

$$\begin{aligned} |f(x) - f_N(x)| &\leqslant \frac{r_N\hat{}\, e^{-x} x^{N+1}}{(N+1)!} \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ &= r_N\hat{}\, \delta_N(x), \\ \delta_N(x) &= \frac{x^{N+1}}{(N+1)!}. \end{aligned} \tag{8}$$

Thus for a given x we will have m decimal accuracy even for the most slowly convergent s_n if N is such that

$$x^{N+1}/(N+1)! < \tfrac{1}{2} \times 10^{-m-1}, \tag{9}$$

N suitably large. The use of Stirling's formula shows that we must have approximately

$$x < \frac{N+1}{e} \left\{ \left(\frac{\pi(N+1)}{2} \right)^{1/2} 10^{-m-1} \right\}^{1/(N+1)}. \tag{10}$$

Table I indicates how large x may be taken for a given accuracy and a given N.

TABLE I

Values of a for Given N and m[a]

	m							
N	3	4	5	6	7	8	9	10
10	2.0	1.6	1.3	1.0	0.8	0.7		
15	3.6	3.1	2.7	2.3	2.0	1.7	1.5	
20	5.4	4.8	4.3	3.8	3.4	3.1	2.8	2.5
30	9.0	8.3	7.7	7.2	6.7	6.2	5.7	5.3
50	16.3	15.6	14.9	14.3	13.6	13.0	12.5	11.9
70	23.7	23.0	22.2	21.5	20.8	20.2	19.5	18.9
100	34.8	34.0	33.2	32.5	31.7	31.0	30.3	29.6

[a] To compute $f(x)$ to m-digit accuracy using $f_N(x)$ take $x < a$.

3. Large x

The sequence s_n often has an asymptotic or convergent representation of the form

$$s_n \sim s + \lambda^n \left[\frac{C_1}{n} + \frac{C_2}{n^2} + \cdots \right], \qquad \lambda \neq 0, |\lambda| \leqslant 1, n \to \infty. \tag{11}$$

In such cases, an asymptotic representation may be obtained for $f(x)$ as $x \to \infty$.

To start, we seek to determine the Borel transform of a simple sequence, $\lambda^n/(n+a)_k$, $a > 0$, $k = 1, 2, \ldots$.

$$\frac{\lambda^n}{(n+a)_k} \leftrightarrow \frac{e^{-x}}{(a)_k} \sum_{n=0}^{\infty} \frac{(x\lambda)^n (a)_n}{n!(a+k)_n} = \frac{e^{-x}}{(a)_k} \Phi(a, a+k, x\lambda) = f^{(k)}(x). \tag{12}$$

For large x, $f^{(k)}(x)$ has the asymptotic behavior

$$f^{(k)}(x) \sim \frac{e^{x(\lambda-1)}}{(x\lambda)^k} \sum_{r=0}^{\infty} \frac{(k)_r (1-a)_r (x\lambda)^{-r}}{r!}$$
$$+ \frac{\Gamma(a)\, e^{-x} \cos(\pi a)}{\Gamma(k)(x\lambda)^a} \sum_{r=0}^{k-1} \frac{(a)_r (1-k)_r}{r!} (-1)^r (x\lambda)^{-r}, \quad x \to \infty, \tag{13}$$

see [3, Vol. I, p. 278]. Note that the second term above is finite (convergent).

The most important case is the case when $a = 1$. Then all terms but the first of the first sum vanish and we have the exact representation

$$\frac{\lambda^n}{(n+1)_k} \leftrightarrow \frac{e^{x(\lambda-1)}}{(x\lambda)^k} + V_k,$$
$$V_k = \frac{-e^{-x}}{\Gamma(k)} \sum_{r=0}^{k-1} (1-k)_r (-1)^r (x\lambda)^{-r-1} = O\left(\frac{e^{-x}}{x}\right), \qquad x \to \infty. \tag{14}$$

We now use the fact that

$$\sum_{s=k}^{\infty} \frac{A_{k,s}}{(n+1)_s} = \frac{1}{n^k}, \qquad n > 0, \tag{15}$$

where the $A_{k,s}$ may be written in terms of the generalized Bernoulli polynomials as follows (see Table II)

$$A_{k,s} = \binom{s-1}{k-1} B_{s-k}^{(s)}(s), \qquad k \leqslant s, s = 1, 2, 3, \ldots. \tag{16}$$

See Nörlund [4, p. 261]. $A_{k,s}$ can be conveniently calculated from

$$A_{k,s} = \text{coefficient of } x^{k-1} \text{ in } (x+1)(x+2)\cdots(x+s-1), \qquad s = 1, 2, 3, \ldots . \tag{17}$$

See [4, p. 147]. Thus

$$\lambda^n/n^k \leftrightarrow e^{x(\lambda-1)} \sum_{s=k}^{\infty} A_{k,s}(x\lambda)^{-k} + U_k$$
$$U_k = O(e^{-x}/x), \tag{18}$$

TABLE II

$A_{k,s}$

	k						
s	1	2	3	4	5	6	7
1	1						
2	1	1					
3	2	3	1				
4	6	11	6	1			
5	24	50	35	10	1		
6	120	274	225	85	15	1	
7	720	1764	1624	735	175	21	1

the series being an asymptotic series. (The above estimates can easily be justified in our result (19)–(20) below by first assuming that s_n is a series of terms $\lambda^n/(n+1)_k$ and then rearranging in terms λ^n/n^k. The present computations seem to be more straightforward.)

Now using representation (11) and invoking the linearity of the Borel transform we find that if

$$s_n \sim s + \lambda^n \left[\frac{C_1}{n} + \frac{C_2}{n^2} + \cdots\right], \qquad n \to \infty, \tag{19}$$

then

$$f(x) = e^{-x} \sum_{n=0}^{\infty} \frac{x^n s_n}{n!} \sim s + e^{x(\lambda-1)} \left[\frac{\bar{C}_1}{x} + \frac{\bar{C}_2}{x^2} + \frac{\bar{C}_3}{x^3} + \cdots\right], \qquad x \to \infty, \tag{20}$$

where

$$\begin{aligned} &\bar{C}_s = \lambda^{-s} \sum_{k=1}^{s} A_{k,s} C_k\,, \\ &\bar{C}_1 = \frac{C_1}{\lambda}, \qquad \bar{C}_2 = \frac{C_1 + C_2}{\lambda^2}, \qquad \bar{C}_3 = \frac{2C_1 + 3C_2 + C_3}{\lambda^3}, \\ &\bar{C}_6 = \frac{6C_1 + 11C_2 + 6C_3 + C_6}{\lambda^4}, \ldots . \end{aligned} \tag{21}$$

When s_n has a known factorial series development

$$s_n = s + \lambda^n \left[\frac{D_1}{(n+1)} + \frac{D_2}{(n+1)(n+2)} + \cdots\right] \tag{22}$$

with, say,

$$\overline{\lim_{k\to\infty}}\ | D_k |^{1/k} < \sigma \tag{23}$$

then all series are convergent and we have

$$\begin{aligned} &s_n \leftrightarrow s + e^{x(\lambda-1)} \sum_{k=1}^{\infty} D_k (x\lambda)^{-k} - e^{-x} \sum_{k=1}^{\infty} D_k^* (x\lambda)^{-k}, \\ &D_k^* = \sum_{r=0}^{\infty} \frac{D_{k+r}}{r!}, \qquad | x\lambda | > \sigma. \end{aligned} \tag{24}$$

An interesting case is the case when $D_k = (-\beta)^k$, $\beta > 0$. Then

$$s_n \leftrightarrow s - \frac{\beta e^{-x}(e^{\lambda x} - e^{-\beta})}{(\beta + x\lambda)}, \tag{25}$$

and setting $\lambda = 1$ gives

$$\Phi(1, n+1, -\beta) \leftrightarrow (x + \beta e^{-x-\beta})/(x + \beta). \tag{26}$$

This Borel transform has a close relationship to some transforms occurring in turbulent scattering theory.

4. Examples

Consider the following incoherent scattering function for a surface with an exponential correlation function:

$$\phi(\alpha, \beta) = e^{-\alpha} \sum_{n=1}^{\infty} \frac{\alpha^n}{n!\, n^2} \left[1 + \frac{\beta}{n^2}\right]^{-3/2},$$
$$\alpha = [k\sigma(\cos\theta_0 + \cos\theta)]^2, \tag{27}$$
$$\beta = [ka(\sin\theta_0 + \sin\theta)]^2$$

where θ_0 is the angle the position vector of the transmitter makes with the vertical, θ the angle the position vector of the receiver makes with the vertical, σ the radar cross section, k the radar wavelength, and a the correlation length of wave.

$$s_n = \frac{[1 + (\beta/n^2)]^{-3/2}}{n^2} = \frac{1}{n^2} - \frac{\frac{3}{2}\beta}{n^4} + \cdots. \tag{28}$$

We have $\lambda = 1$, $s = 0$, and

$$C_1 = C_3 = C_5 = \cdots = 0,$$
$$C_2 = 1, \qquad C_4 = \frac{-3}{2}\beta, \qquad C_6 = \frac{15}{8}\beta^2, \qquad C_8 = \frac{-35}{16}\beta^3, \ldots. \tag{29}$$

Thus

$$\phi(\alpha, \beta) \sim \frac{1}{\alpha^2} + \frac{3}{\alpha^3} + \frac{11 - \frac{3}{2}\beta}{\alpha^4} + \frac{50 - 15\beta}{\alpha^5} + \cdots, \qquad \alpha \to \infty. \tag{30}$$

For $\alpha = 10$, $\beta = 1$ the terms above give 0.01430 with an error 2×10^{-5}. Notice the expansion is not uniform in β and the accuracy deteriorates with increasing β. In any case, a good policy for computing from asymptotic expansions is to stop before the smallest term; see Knopp [2].

Next consider the incoherent scattering function for a surface with a Gaussian correlation function

$$\Psi(\alpha, \beta) = e^{-\alpha} \sum_{n=1}^{\infty} \frac{\alpha^n}{n!} \left[\frac{e^{-\beta/n}}{n}\right], \tag{31}$$

where α, β are as before except a is to be replaced by $a/2$ in (27). Then

$$s_n = \frac{e^{-\beta/n}}{n} = \frac{1}{n} - \frac{\beta}{n^2} + \frac{\beta^2/2}{n^3} - \frac{\beta^3/6}{n^4} + \cdots, \tag{32}$$

so again $\lambda = 1$ and

$$\Psi(\alpha, \beta) \sim \frac{1}{\alpha} + \frac{1-\beta}{\alpha^2} + \frac{(\beta^2 - 6\beta + 4)}{2\alpha^3} - \frac{(\beta^3 - 18\beta^2 + 66\beta - 36)}{6\alpha^4} + \cdots, \qquad \alpha \to \infty. \tag{33}$$

With $\beta = \frac{1}{2}$, $\alpha = 10$ the four terms above give $\Psi = 0.1057$ with an error of less than one-half unit in the last decimal place.

5. Comments

The transform pair given in Section 2,

$$s_n = \Phi(1, n+1; -\beta) \leftrightarrow (x + \beta e^{-x-\beta})/(x+\beta) = f(x), \tag{34}$$

has some of the characteristics of the Gaussian correlation transform pair

$$t_n = e^{-\beta/n}/n \leftrightarrow g(x). \tag{35}$$

For β large and $x \ll \beta$, f and g are exponentially small in x. Nevertheless, f and g ultimately behave algebraically in x, $f = 1 + o(1)$, $g = (1/x)[1 + o(1)]$ as $x \to \infty$. Thus there is a transitional x-region in which f and g move from exponential behavior to algebraic behavior.

The graph of g given in [1] reflects this, the graph becoming increasingly steep as β increases in the neighborhood of $x = 10$.

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NEW METHODS FOR ACCELERATING THE CONVERGENCE OF SEQUENCES ARISING IN LAPLACE TRANSFORM THEORY*

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Abstract. We discuss a new class of linear summation methods which are highly effective in accelerating the convergence of sequences which can be represented by Laplace moment integrals with generating function $f(t)$. Different formulas are obtained corresponding to different L_p^r classes for f.

The methods are shown to be nonregular, but for the sequences studied they do preserve convergence and are generally much more efficient than the usual regular methods. We provide simple error bounds.

We close with criteria which should help the reader to decide when a given sequence is a likely candidate for a successful application of these methods.

Introduction. Many methods have been devised to accelerate the convergence of numerical sequences. (For a survey of these, see the article [1].) These methods, generally called summation methods, transform the given sequence s_n, $n = 0, 1, 2, \cdots$, into a new sequence $\bar{s}_n$

$$T(s_n) = \bar{s}_n$$

which, one hopes, will converge to the same limit as s_n but more rapidly.

Methods which preserve convergence

$$s_n \to \xi \Rightarrow \bar{s}_n \to \xi$$

are called regular methods; otherwise, the methods are called nonregular.

The method is linear if

$$T(c_1 s_n + c_2 t_n) = c_1 \bar{s}_n + c_2 \bar{t}_n.$$

Sometimes the sequence s_n is mapped into a double sequence

$$T(s_n) = \bar{s}_{n,m}, \qquad n, m = 0, 1, 2, \cdots.$$

Convergence in the double array $\bar{s}_{n,m}$ can then be examined as one moves out and/or down a path in the array, i.e., as $m + n \to \infty$. Some paths produce more rapid convergence than others. Such methods may be nonlinear or linear, see [1], [2].

The classical methods (treated, for instance, in Knopp [3, p. 391ff, p. 457ff]) are linear. More recently, nonlinear methods have been proposed, for example, Shanks [4], Wynn [5], (whose formalization of the Shanks transformation is called the ε-algorithm)[1] and Wimp [1, p. 203]. These methods are nonregular.

It must not be thought, however, that nonregularity in a summation method is always a shortcoming. Nonregular methods often work so well—i.e., accelerate convergence so dramatically—on the sequences for which they do preserve convergence that one is willing to forego the convenience of using a method which

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[1] This author has developed the theory of the ε algorithm in an interesting series of subsequent papers.

works for every convergent sequence. Clearly, using a nonregular method can be risky unless one has an a priori method of deciding whether the method will preserve convergence of the given sequence.

The class of methods discussed in this paper are nonregular. They are also linear. A special case is known and has been discussed by several authors, including the present one (again, see [1]). Up until now, no one has furnished a very practical way of deciding for which sequences this method will preserve convergence or how much convergence is improved. In this paper we do two things. We derive an entire class of new methods—all nonregular but each having different computational advantages—and we show that when the given sequence has a certain representation as a Laplace-type moment integral, the methods will both preserve and enhance convergence. Also, we provide objective criteria for measuring the improvement in convergence.

We close with some examples and general observations which should help the reader to decide when these methods should be chosen over other known methods.

1. Derivation of the methods. In our methods, the transformed sequence, $\bar{s}_n$, will be defined by taking a weighted average of the terms $s_0, s_1, \cdots, s_n$ of the given sequence. (Such methods are sometimes called Toeplitz methods.) To do this we choose a double array of weights $\{\mu_{n,k}\}$, $n=0, 1, 2, \cdots, k=0, 1, \cdots, n$, and compose a matrix

$$U=\begin{bmatrix} \mu_{00} & 0 & 0 & \cdots \\ \mu_{10} & \mu_{11} & 0 & \cdots \\ \mu_{20} & \mu_{21} & \mu_{22} & \cdots \end{bmatrix}, \tag{1.1}$$

with the weights chosen so that the rows sum to 1, i.e.,

$$\sum_{k=0}^{n} \mu_{n,k}=1. \tag{1.2}$$

We then define

$$\bar{s}_n=\sum_{k=0}^{n} \mu_{n,k}s_k, \qquad n=0, 1, 2, \cdots. \tag{1.3}$$

We predicate our derivation on the assumption that s_n may be represented as a constant ξ plus a certain moment sequence as follows:

$$s_n=\xi+\bar{f}(\alpha+n) \tag{1.4}$$

for some Re $\alpha>0$, $n=0, 1, 2, \cdots$, where $f\in L_1(0, \infty)$ and

$$\bar{f}(p)=\int_0^\infty e^{-pt}f(t)\,dt. \tag{1.5}$$

(The motivation for representing s_n this way will be provided later.)

We call the polynomial

$$p_n(t)=\sum_{k=0}^{n} \mu_{n,k}t^k \tag{1.6}$$

the nth *U-associated polynomial.* (Note $p_n(1)=1$). Since, under the given assumption, $s_n \to \xi$, the error of the sequence s_n is

$$r_n = s_n - \xi = f(\alpha+n), \tag{1.7}$$

and the error of the transformed sequence is

$$\bar{r}_n = \sum_{k=0}^{n} \mu_{n,k} r_k = \bar{s}_n - \xi \tag{1.8}$$

by virtue of the normalization (1.2).

We next prove

THEOREM 1. *Let $f \in L_1(0,\infty)$, $f^{(r)}$ be locally integrable and*

$$f^{(r)}(u)\, e^{-\alpha c_1 u} \in L_{q_1}(0,\infty) \qquad q_1>1, \quad 0<c_1<1. \tag{1.9}$$

Further, let the U-associated polynomials $p_n(t)$ satisfy the r conditions

$$\int_0^\infty e^{-\alpha t} p_n(e^{-t}) t^j \, dt = 0, \qquad 0 \leqq j \leqq r-1. \tag{1.10}$$

Then the error of the transformed sequence has the bound

$$|\bar{r}_n| \leqq \|f^{(r)}(u)\, e^{-\alpha c_1 u}\|_{q_1} E_n^{q_2}, \tag{1.11}$$

where

$$E_n^{q_2} = \left\| \sum_{k=0}^{n} \frac{\mu_{n,k}\, e^{-\alpha c_2 u - ku}}{(\alpha+k)^r} \right\|_{q_2}, \qquad c_1+c_2=1, \quad \frac{1}{q_1}+\frac{1}{q_2}=1. \tag{1.12}$$

Equality in (1.11) *is attained for some f in the class of functions* (1.9).

Proof. f may be represented

$$f(t) = \sum_{i=0}^{r-1} \frac{f^{(i)}(0)t^i}{i!} + \frac{1}{\Gamma(r)} \int_0^\infty f^{(r)}(u)(t-u)^{r-1} E(t-u)\, du, \tag{1.13}$$

where

$$E(t) = \begin{cases} 1, & t>0, \\ 0, & t<0. \end{cases} \tag{1.14}$$

Substituting this in the representation (1.4)–(1.5), summing according to (1.3), (1.6) and (1.7) and using (1.10) we find, after an easily justified change in the order of integration,

$$\bar{r}_n = \int_0^\infty f^{(r)}(u) K_n(u)\, du \tag{1.15}$$

where

$$K_n(u) = \frac{1}{\Gamma(r)} \int_u^\infty e^{-\alpha t} p_n(e^{-t})(t-u)^{r-1}\, dt. \tag{1.16}$$

Now, let $t-u=x$:

$$K_n(u) = e^{-\alpha u} \sum_{k=0}^{n} \frac{\mu_{n,k}\, e^{-ku}}{(\alpha+k)^r} = O(e^{-\alpha u}), \qquad u \to \infty. \tag{1.17}$$

works for every convergent sequence. Clearly, using a nonregular method can be risky unless one has an a priori method of deciding whether the method will preserve convergence of the given sequence.

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Now, let $t-u=x$:

$$K_n(u) = e^{-\alpha u} \sum_{k=0}^{n} \frac{\mu_{n,k}\, e^{-ku}}{(\alpha+k)^r} = O(e^{-\alpha u}), \qquad u \to \infty. \tag{1.17}$$

Because of the integrability condition (1.9), Hölder's inequality may be invoked and we have the theorem. (That an f may be found yielding equality is a standard argument, see [6, p. 134ff].)

It is desirable to use a more workable representation of $K_n(u)$ and thus of $E_n^{q_2}$. Define the polynomial

$$p_{n,r}(t)=\sum_{k=0}^{n}\frac{\mu_{n,k}t^k}{(\alpha+k)^r} \tag{1.18}$$

so

$$K_n(u)=e^{-\alpha u}p_{n,r}(e^{-u}). \tag{1.19}$$

We need a

LEMMA. *$p_{n,r}(t)$ has the representation*

$$p_{n,r}(t)=(1-t)^r\tau_{n,r}(t) \tag{1.20}$$

where $\tau_{n,r}(t)$ is a polynomial of degree $n-r$ satisfying

$$\tau_{n,r}(1)=\frac{(-1)^r}{r!}. \tag{1.21}$$

Proof. Condition (1.10) becomes

$$\int_0^\infty e^{-\alpha t}\sum_{k=0}^{n}e^{-kt}\mu_{n,k}t^j\,dt=\sum_{k=0}^{n}\frac{\mu_{n,k}}{(\alpha+k)^{j+1}}=0,\qquad 0\leqq j\leqq r-1, \tag{1.22}$$

or

$$\frac{\partial^s}{\partial t^s}\sum_{k=0}^{n}\frac{\mu_{n,k}\,e^{-t(\alpha+k)}}{(\alpha+k)^r}\bigg|_{t=0}=0,\qquad 0\leqq s\leqq r-1. \tag{1.23}$$

But condition (1.23) implies

$$\frac{\partial^s}{\partial t^s}e^{-\alpha t}p_{n,r}(e^{-t})\bigg|_{t=0}=0,\qquad 0\leqq s\leqq r-1. \tag{1.24}$$

so

$$p_{n,r}(t)=(1-t)^r\tau_{n,r}(t). \tag{1.25}$$

We also must have

$$\frac{\partial^r}{\partial t^r}e^{-\alpha t}p_{n,r}(e^{-t})=(-1)^r e^{-\alpha t}\sum_{k=0}^{n}\mu_{n,k}\,e^{-kt}=(-1)^r e^{-\alpha t}p_n(e^{-t}). \tag{1.26}$$

We next calculate $\tau_{n,r}(1)$.

$$\begin{aligned}\frac{\partial^r}{\partial t^r}e^{-\alpha t}p_{n,r}(e^{-t})\bigg|_{t=0}&=(-1)^r=\frac{\partial^r}{\partial t^r}\{e^{-t}(1-e^{-t})^r\tau_{n,r}(e^{-t})\}\bigg|_{t=0}\\ &=\sum_{m=0}^{r}\binom{r}{m}\frac{\partial^{r-m}}{\partial t^{r-m}}(1-e^{-t})^r\frac{\partial^m}{\partial t^m}\\ &\quad\cdot\{e^{-\alpha t}\tau_{n,r}(e^{-t})\}\bigg|_{t=0}\end{aligned} \tag{1.27}$$

or

$$r!\, e^{-\alpha t}\tau_{n,r}(e^{-t})|_{t=0} = (-1)^r \tag{1.28}$$

and

$$\tau_{n,r}(1) = \frac{(-1)^r}{r!}. \tag{1.29}$$

Thus we have the desired form

$$\begin{aligned} E_n^{q_2} &= \|e^{\alpha c_1 u} K_n(u)\|_{q_2} = \|e^{-\alpha c_2 u} p_{n,r}(e^{-u})\|_{q_2} \\ &= \left| \int_0^1 x^{\alpha c_2 q_2 - 1}(1-x)^{r q_2} |\tau_{n,r}(x)|^{q_2}\, dx \right|^{(1/q_2)}. \end{aligned} \tag{1.30}$$

There are two cases in which the minimization of $E_n^{q_2}$ can be easily effected. Performing the minimization will yield the weights $\{\mu_{n,k}\}$.

2. The case $q_1 = q_2 = 2$. The problem is to choose $\tau_{n,r}$ to minimize (1.30), i.e., to find

$$\min_{\tau_{n,r}} \int_0^1 x^{2\alpha c_2 - 1}(1-x)^{2r}\tau_{n,r}^2(x)\, dx = M_n \tag{2.1}$$

subject to condition (1.29). This minimum problem is associated with the Jacobi polynomials. Let $R_n(x) = P_n^{(2r+1, 2\alpha c_2 - 1)}(2x-1)$ be the shifted Jacobi polynomial in the usual standardization [7]. The solution to (2.1), fully worked out in [8], is

$$\tau_{n,r}(x) = \frac{(-1)^r (n-r)!(2r+1)!}{r!(n+r+1)!} R_{n-r}(x), \tag{2.2}$$

$$\begin{aligned} M_n &= \frac{\Gamma(n-r+2\alpha c_2)(n-r)!(2r)!(2r+1)!}{\Gamma(n+r+2\alpha c_2+1)(n+r+1)!r!^2} \\ &= \frac{(2r!)^2(2r+1)n^{-4r-2}}{r!^2}[1+O(n^{-1})], \qquad n\to\infty. \end{aligned} \tag{2.3}$$

If we use the notation

$$\delta_k f = \{\text{coefficient of } t^k \text{ in } f, f \text{ a polynomial}\} \tag{2.4}$$

we have an explicit formula for the entries in U

$$\mu_{n,k} = \frac{(\alpha+k)^r(-1)^r(n-r)!(2r+1)!}{r!(n+r+1)!}\delta_k[(1-t)^r R_{n-r}(t)]. \tag{2.5}$$

Equation (2.3) gives

$$|\bar{r}_n| \le \|f^{(r)}(u)\, e^{-\alpha c_1 u}\|_2 \frac{(2r)!}{r!}\sqrt{2r+1}\, n^{-2r-1}\left(1+\frac{A}{n}\right), \tag{2.6}$$

$n > n_0$, for some A independent of f, equality being attained for some f in the class (1.9).

It is useful to have an explicit representation of $p_n(t)$. Let

$$\nu_{n,k} = \frac{(-1)^r(n-r)!(2r+1)!}{r!(n+r+1)!}\delta_k[(1-t)^r R_{n-r}(t)]. \tag{2.7}$$

Then

$$\sum_{k=0}^{n} \nu_{n,k}t^k = q_n(t) = \frac{(-1)^r(n-r)!(2r+1)!}{r!(n+r+1)!}(1-t)^r R_{n-r}(t), \tag{2.8}$$

$$p_n(t) = t^{-\alpha}(tD)^r[t^\alpha q_n(t)], \tag{2.9}$$

or

$$p_n(t) = \frac{t^{-\alpha}(-1)^r(n-r)!(2r+1)!}{r!(n+r+1)!}(tD)^r\{t^\alpha(1-t)^r R_{n-r}(t)\}. \tag{2.10}$$

Since all the zeros of R_n are simple and in (0, 1) (in fact, dense in (0, 1) as $n \to \infty$) an application of Rolle's theorem shows $p_n(t)$ has all its zeros in (0, 1) (and dense there as $n \to \infty$.)

That the summation method above is not regular is a consequence of the following

THEOREM 2. *Let*

$$p_n(t) = \sum_{k=0}^{n} \mu_{n,k}t^k, \qquad p_n(1) = 1 \tag{2.11}$$

have all its zeros in $[0, \infty]$ *for n sufficiently large with* m_n *of its zeros bounded and bounded away from zero,* $m_n \to \infty$ *as* $n \to \infty$. *Then the method* $U = [\mu_{n,k}]$ *is not regular.*

Proof. Write

$$p_n(t) = \prod_{j=1}^{n}\left(\frac{t-a_j}{1-a_j}\right), \qquad a_j \geqq 0. \tag{2.12}$$

Then

$$A_n = \sum_{k=0}^{n} |\mu_{n,k}| = \prod_{j=1}^{n}\left|\frac{1+a_j}{1-a_j}\right|, \qquad n > n_0. \tag{2.13}$$

For these m_n zeros in $[\varepsilon, 1/\varepsilon]$, $0 < \varepsilon < 1$, we have

$$\left|\frac{1+a_j}{1-a_j}\right| \geqq 1+\delta, \qquad \delta > 0 \tag{2.14}$$

so

$$|A_n| \geqq (1+\delta)^{m_n} \to \infty \quad \text{as } n \to \infty. \tag{2.15}$$

By Hardy's theorem [9, p. 43], U is not regular.

However, for most sequences of the form (1.4), the improvement in convergence is considerable.

THEOREM 3. *Let f be as in* (1.5), (1.9). *Let there be a least one j, $0 \leqq j \leqq r-1$, for which $f^{(j)}(0) \neq 0$. Then*

$$\bar{r}_n = n^{j-2r-3/2} O(r_n), \qquad n \to \infty. \tag{2.16}$$

Proof. We find

$$K_n(u) = \frac{(-1)^r (n-r)!(2r+1)!}{r!(n+r+1)!} (1-e^{-u})^r e^{-\alpha u} R_{n-r}(e^{-u}) \tag{2.17}$$

so

$$\begin{aligned} \bar{r}_n &= \frac{(2r+1)!(-1)^r n^{-2r-1}}{r!} \left[1 + O\left(\frac{1}{n}\right)\right] \\ &\quad \cdot \int_0^\infty e^{-\alpha u} (1-e^{-u})^r R_{n-r}(e^{-u}) f^{(r)}(u)\, du \\ &= n^{-2r-3/2} O(1), \qquad n \to \infty. \end{aligned} \tag{2.18}$$

by the use of [7, vol. II, p. 206].

Also, we have

$$\begin{aligned} r_n &= \frac{f(0)}{\alpha+n} + \cdots + \frac{f^{(r-1)}(0)}{(\alpha+n)^r} + \frac{1}{(\alpha+n)^r} \int_0^\infty e^{-(\alpha+n)t} f^{(r)}(t)\, dt \\ &= \frac{f^{(j)}(0)}{(\alpha+n)^j} + \cdots + \frac{f^{(r-1)}(0)}{(\alpha+n)^r} + \frac{o(1)}{(\alpha+n)^r}. \end{aligned} \tag{2.19}$$

Combining these two estimates gives the theorem.

To get a clearer idea of the computational properties of the algorithm, we will derive an asymptotic estimate for $\mu_{n,k}$. Let $\beta = 2\alpha c_2 - 1$. Taking the Cauchy product of the series for $(1-t)^r$ and $R_{n-r}(t)$ and selecting the coefficient of t^k gives

$$\begin{aligned} &\delta_k\{(1-t)^r R_{n-r}(t)\} \\ &\quad = \frac{(-1)^{n+r+k+1} \Gamma(n-r+\beta+1)(n+r+\beta+2)_k}{\Gamma(\beta+k+1)\Gamma(n-r+1-k)k!} \\ &\qquad \cdot {}_3F_2\left(\left. \begin{matrix} -r, -k, -k-\beta \\ -n-r-k-\beta-1, n-r+1-k \end{matrix} \right| 1 \right). \end{aligned} \tag{2.20}$$

If $0 \leqq k \leqq n-r$ each term in the ${}_3F_2$ on the right is positive. Thus the ${}_3F_2$ is greater than the first term $(=1)$ and less than the ${}_3F_2$ resulting if k is replaced by $k+n+r+1$ in the third numerator parameter, i.e.,

$$1 \leqq {}_3F_2 \leqq {}_2F_1\left(\left. \begin{matrix} -r, \quad -k \\ n+1-r-k \end{matrix} \right| 1 \right) = \frac{\Gamma(n+1-r-k)\Gamma(n+1)}{\Gamma(n+1-k)\Gamma(n+1-r)} \approx 1, \qquad n \to \infty. \tag{2.21}$$

Thus for *fixed* k,

$$\mu_{n,k} = \frac{(\alpha+k)^r (2r+1)!(-1)^{n+k+1}}{r!k!\Gamma(\beta+k+1)} n^{2k+\beta-2r-1} \phi_n, \tag{2.22}$$

where $\phi_n \to 1$ as $n \to \infty$ and $\phi_n \geqq 1$. This formula, of course, confirms the nonregularity of the methods. Further, the $\mu_{n,k}$ get quite large and alternate in sign with k. Using the methods requires judgment.

As a simple application of the previous methods, take

$$s_n \equiv s_n(z) = \sum_{k=1}^{n+1} k^{-z}, \tag{2.23}$$

$$r_n(z) = \zeta(z) - s_n(z), \qquad \operatorname{Re} z > 1. \tag{2.24}$$

We have

$$r_n(z) = \frac{1}{\Gamma(z)} \int_0^\infty e^{-nt} f(t)\, dt, \tag{2.25}$$

$$f(t) = \frac{e^{-t} t^{z-1}}{e^t - 1} = O(t^{z-2}), \qquad t \to 0^+. \tag{2.26}$$

Take $z = 2.8$,

$$\zeta(2.8) = 1.247031 \cdots. \tag{2.27}$$

For $r = 1$, $n = 5$, $\alpha = 1$, $c_1 = c_2 = \frac{1}{2}$ we find

$$\{\mu_{n,k}\} = \tfrac{1}{35}\{-1, 66, -744, 2784, -4050, 1980\}, \tag{2.28}$$

$$\begin{aligned} \bar{s}_5 &= 1.250704 \cdots, \qquad \bar{r}_5 \sim -3.7 \times 10^{-3}, \\ s_5 &= 1.228005 \cdots, \qquad r_5 \sim 1.9 \times 10^{-2}. \end{aligned} \tag{2.29}$$

We have chosen an f in (2.22) with a modest order of differentiability ($r = 1$). In such cases, the improvement in convergence can only be modest, i.e. algebraic, in accordance with the estimate given by (2.6).

All the methods (2.5) work better when a function f is chosen which has a higher order of differentiability than that for which the formula (2.5) was derived. For instance, if

$$s_n = \sum_{k=1}^{n+1} \frac{1}{k} - \ln(n+1) = \gamma + \int_0^\infty e^{-nt} f(t)\, dt \tag{2.30}$$

then

$$\begin{aligned} f(t) &= e^{-t}[t^{-1} - (e^t - 1)^{-1}] \in C^\infty[0, \infty], \\ \gamma &= .57721\,56649 \cdots \text{ (Euler's constant)} \end{aligned} \tag{2.31}$$

and if $\bar{s}_5$ is calculated using the weights with $r = 3$, $\alpha = 1$, $n = 5$, $c_1 = c_2 = \frac{1}{2}$, we have

$$\bar{s}_5 = .577209 \cdots, \ \bar{r}_5 \sim 6.1 \times 10^{-6}. \tag{2.32}$$

We know of no other linear method which will produce such a spectacular improvement in convergence in the sequence (2.30). ($s_5 = .658 \cdots$ is not accurate to one significant figure.)

A recursion relation may be obtained for the $\mu_{n,k}$ by using the known recursion relation for the Jacobi polynomials [7, vol. II, p. 169]. If one observes that

$$\partial_k[(1-t)^r t R_{n-r}(t)] = \partial_{k-1}[(1-t)^r R_{n-r}(t)] \tag{2.33}$$

then the computations are straightforward. Multiply the recursion formula by $(1-t)^r$ after replacing x by $2t-1$ and perform ∂_k on each term to find

$$\begin{gathered} A_n\mu_{n+1,k} = B_n\mu_{n,k} + C_n\mu_{n,k-1} + D_n\mu_{n-1,k} \\ 0 \leqq k \leqq n+1, \qquad n > r, \end{gathered} \tag{2.34}$$

with the interpretation $\mu_{n,-1} = \mu_{n,n+1} = \mu_{n,n+2} = 0$ and where

$$\begin{aligned} A_n &= (n+r+2)(n+r+2\alpha c_2+1)(n+\alpha c_2), \\ B_n &= (2n+2\alpha c_2+1)[(r+\alpha c_2)(r+1-\alpha c_2)-(n+\alpha c_2)(n+\alpha c_2+1)], \\ C_n &= 2(2n+2\alpha c_2+1)(n+\alpha c_2)(n+\alpha c_2+1)\left(\frac{\alpha+k}{\alpha+k-1}\right)^r, \\ D_n &= -(n-r)(n-r+2\alpha c_2-1)(n+\alpha c_2+1). \end{aligned} \tag{2.35}$$

To start the computations, one needs

$$\begin{aligned} \mu_{r,k} &= \frac{(\alpha+k)^r(-1)^{r+k}}{k!(r-k)!}, \qquad 0 \leqq k \leqq r, \\ \mu_{r+1,k} &= \frac{(\alpha+k)^r(-1)^{r+k+1}(k+\alpha c_2)}{k!(r-k+1)!}, \qquad 0 \leqq k \leqq r+1. \end{aligned} \tag{2.36}$$

(The presence of the factor $[(\alpha+k)/(\alpha+k-1)]^r$ in C_n precludes the neat rhombus-rule formalization possible for the methods discussed in [2].)

3. The case $r = \infty$. The requirement (1.10) becomes

$$\int_0^\infty e^{-\alpha t} p_n(e^{-t}) t^j \, dt = 0, \qquad 0 \leqq j \leqq n-1, \tag{3.1}$$

or

$$\sum_{k=0}^{n} \frac{\mu_{n,k}}{(\alpha+k)^{j+1}} = 0, \qquad 0 \leqq j \leqq n-1. \tag{3.2}$$

We have

$$\mu_{n,k} = \binom{n}{k}(-1)^{k+n}\frac{(\alpha+k)^n}{n!} \tag{3.3}$$

for substituting this value in (3.2) yields

$$\begin{aligned} \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{k+n}(\alpha+k)^{n-j-1} &= \frac{1}{n!}\Delta^n \alpha^{n-j-1} \\ &= \sum_{k=0}^{n}\frac{\mu_{n,k}}{(\alpha+k)^{j+1}} \\ &= \begin{cases} 0, & 0 \leqq j \leqq n-1, \\ 1, & j = -1. \end{cases} \end{aligned} \tag{3.4}$$

where $\phi_n \to 1$ as $n \to \infty$ and $\phi_n \geqq 1$. This formula, of course, confirms the nonregularity of the methods. Further, the $\mu_{n,k}$ get quite large and alternate in sign with k. Using the methods requires judgment.

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We have

$$\mu_{n,k} = \binom{n}{k}(-1)^{k+n}\frac{(\alpha+k)^n}{n!} \tag{3.3}$$

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$$\begin{aligned} \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(-1)^{k+n}(\alpha+k)^{n-j-1} &= \frac{1}{n!}\Delta^n \alpha^{n-j-1} \\ &= \sum_{k=0}^{n}\frac{\mu_{n,k}}{(\alpha+k)^{j+1}} \\ &= \begin{cases} 0, & 0 \leqq j \leqq n-1, \\ 1, & j = -1. \end{cases} \end{aligned} \tag{3.4}$$

For the error, we have

$$|\bar{r}_n| \leqq \|f^{(n)}(u)\, e^{-\alpha c_1 u}\|_{q_1} \| \sum_{k=0}^{n} \frac{\mu_{n,k}\, e^{-\alpha c_2 u - ku}}{(\alpha+k)^n} \|_{q_2}$$

$$(3.5) \qquad = \|f^{(n)}(u)\, e^{-\alpha c_1 u}\|_{q_1} \| \frac{(-1)^n}{n!} e^{-\alpha c_2 u}(1-e^{-u})^n \|_{q_2}$$

$$= \|f^{(n)}(u)\, e^{-\alpha c_1 u}\|_{q_1} \frac{1}{n!} \left\{ \frac{\Gamma(\alpha c_2 q_2)\Gamma(nq_2+1)}{\Gamma(nq_2+1+\alpha c_2 q_2)} \right\}^{1/q_2},$$

$$(3.6) \qquad |\bar{r}_n| \leqq \|f^{(n)}(u)\, e^{-\alpha c_1 u}\|_{q_1} \frac{[\Gamma(\alpha c_2 q_2)]^{1/q_2}}{n!} (nq_2)^{-\alpha c_2} \left(1+\frac{B}{n}\right),$$

$n > n_0$, for some B independent of f, equality being attained in the next to last equation for some $f \in C^{\infty}[0, \infty)$.

This is the same method discussed in [10], where it was shown to be nonregular.

4. Conclusions. In practice one often encounters sequences having the asymptotic behavior

$$(4.1) \qquad s_n \sim \alpha + \sum_{r=0}^{\infty} \frac{d_r}{n^{\alpha+r+1}}, \qquad n \to \infty.$$

If Re $\alpha > -1$ and

$$(4.2) \qquad \overline{\lim_{r\to\infty}} \frac{|d_r|^{(1/r)}}{r} < \infty$$

an f may be determined such that $t^{-\alpha} f(t)$ has a power series convergent in some interval $[0, \delta)$, $f(t) = O(e^{\mu t})$, $t \to \infty$; thus, s_n will have a representation

$$(4.3) \qquad s_n = \alpha + \bar{f}(n).$$

In fact, the two sequences treated in § 2 satisfied these conditions, and provided the motivation for picking the form (1.4)–(1.5) for s_n. Also, such sequences are usually intractable to the traditional regular methods of summation, see [1]. It is easy to demonstrate functions of the class (1.9) for which earlier the Cesaro weights

$$(4.4) \qquad \mu_{n,k} = \frac{1}{n+1}, \qquad k = 0, 1, 2, \cdots, n.$$

or the binomial weights

$$(4.5) \qquad \mu_{n,k} = \binom{n}{k} 2^{-n}, \qquad k = 0, 1, 2, \cdots, n,$$

will produce error sequences with

$$(4.6) \qquad r_n = \frac{A}{n}[1 + o(1)], \qquad n \to \infty.$$

Inequality (2.6) shows the methods developed here are superior. Conversely, when the original sequence is amenable to regular methods, the methods of this paper are probably not the ones to use.

It has been our experience that the nonlinear methods discussed in the survey [1], methods due to Shanks or to the present author, do not perform well on sequences which behave like (4.1)–(4.2). The reasons for this remain to be explored. Levin in [11] has discussed interesting generalizations of the Shanks transformations which are quite effective on many such sequences, for instance, on

$$s_n = \sum_{k=1}^{n+1} \frac{1}{k^2}.$$

However, given such a sequence there is no a priori method known for guaranteeing that convergence is enhanced (or even preserved). None of the nonlinear methods provides an error estimate as convenient as (2.16).

How is one to recognize when the given sequence acts like (4.1)? Well, one criterion is total monotonicity, see [12]. (Such a property, however, is very difficult to verify.) Often s_n is known to satisfy a linear difference equation

$$\sum_{k=0}^{n} a_{n,k} s_{n+k} = v_n, \qquad n = 0, 1, 2, \cdots,$$

whose coefficients $a_{n,k}$ and v_n can be represented as Poincaré type asymptotic series in $1/n$. In certain cases, the Birkhoff–Trjitzinsky asymptotic theory of linear difference equations will then guarantee that s_n behaves like (4.1). The sequence (2.30) is a case in point, for we have

$$s_{n+1} - s_n = \frac{1}{n+2} - \ln\left(1 + \frac{1}{n+1}\right) = \frac{-\frac{1}{2}}{n^2} + \frac{\frac{5}{3}}{n^3} - \frac{\frac{17}{4}}{n^4} + \cdots, \qquad n \to \infty.$$

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THE -METHOD AND FREDHOLM INTEGRAL EQUATIONS *

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Instead of using approximate methods on the equation

$$f(x) = g(x) + \lambda \int_0^1 K(x,t)\, f(t)\, \mathrm{d}t\,,$$

the τ-method is employed to obtain the exact solution of the equation

$$h(x) = g(x) + \lambda \int_0^1 K(x,t)\, h(t)\, \mathrm{d}t + R(x,\lambda)\,.$$

The analytical form of $R(x, \lambda)$ determines the type of approximation which results. In this paper $R(x, \lambda)$ is chosen to be a function which is rational in the parameter λ in which case, the approximation to the true solution has the same character. An example is given.

1. Introduction

Suppose we have the equation

$$L[y(x)] = g(x) \qquad x \in (a, b)\,, \tag{1.1}$$

(where L is a linear differential operator of order p, and $g(x)$ is a given function) along with the appropriate boundary conditions. The philosophy of the τ-method avoids trying to solve this equation directly, even approximately. Rather, we append to the right hand side of the equation a term $\tau_n R_n(x)$, where τ_n is a constant, R_n a polynomial of degree n, and then try to solve the related equation

$$L(y_n(x)] = g(x) + \tau_n R_n(x)\,. \tag{1.2}$$

This seems to be hopelessly complicating the problem, but in fact if L has polynomial coefficients, appending such a term or a linear combination of such terms often enables us to solve the equation exactly and even to obtain a polynomial or rational solution which satisfies the boundary conditions given for equation (1.1). We hope, then, that the sequence of polynomials $y_n(x)$ so determined approaches $y(x)$ rapidly enough as $n \to \infty$ to provide an efficient algorithm for tabulat-

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ing the desired solution of (1.1). This is usually the case if $R_n(x)$ is chosen so that all its zeros lie in (a, b). But at any rate, the beauty of the τ-method is that it provides an elegant equation for the error $\epsilon_n(x) = y(x) - y_n(x)$:

$$L[\epsilon_n(x)] = -\tau_n R_n(x) . \tag{1.3}$$

Often this equation is easy to analyze. For many examples and applications of the τ-method the reader should consult the references [1] and [2].

In this paper we seek to apply the τ-method to the Fredholm integral equation

$$f(x) = g(x) + \lambda \int_0^1 K(x, t) f(t)\, \mathrm{d}t , \tag{1.4}$$

(see [3], [4], [5]). The modification of the method which is required to treat this equation is far from obvious. For instance, the appended term must now depend on two variables, λ and x. The modification we propose here provides approximations which are rational in λ and whose coefficients depend on the iterated kernels of the Fredholm operator. Again, the method provides an elegant error analysis, and the numerical examples we have chosen indicate the algorithm has great power.

In general, our methods do not yield polynomial approximations to the solution of (1.4). For a description of an analog of the τ-method which yields polynomial approximations the reader should consult the book by Fox and Parker [6].

2. Non-interpolatory rational approximations

The equation to be treated is

$$f(x) = g(x) + \lambda \int_0^1 K(x, t) f(t)\, \mathrm{d}t . \tag{2.1}$$

Consider the related equation

$$f_n(x) = g(x) + \lambda \int_0^1 K(x, t) f_n(t)\, \mathrm{d}t + h_n(\lambda, x) , \tag{2.2}$$

where

$$f_n(x) = \frac{A_n(x, \lambda)}{B_{n+1}(\lambda)} , \qquad h_n(\lambda, x) = \frac{\lambda^{n+1} C_n(x)}{B_{n+1}(\lambda)} ,$$

$$\begin{aligned} A_n(x, \lambda) &= \alpha_0(x) + \alpha_1(x)\lambda + ... + \alpha_n(x)\lambda^n , \\ B_{n+1}(\lambda) &= 1 + \beta_1 \lambda + \beta_2 \lambda^2 + ... + \beta_{n+1} \lambda^{n+1} , \end{aligned} \tag{2.3}$$

and the β_k are independent of x.

In order to determine $\alpha_k(x)$ and β_k, we multiply (2.2) by $B_{n+1}(\lambda)$ and equate powers of λ, so

$$\begin{aligned} &\alpha_k(x) = g(x)\,\beta_k + \int_0^1 K(x,t)\,\alpha_{k-1}(t)\,\mathrm{d}t\,, \qquad 1 \leqslant k \leqslant n\,, \\ &\alpha_0(x) = g(x)\,. \end{aligned} \tag{2.4}$$

If we now require that the coefficient of λ^{n+1} be zero, the result is

$$0 = \beta_{n+1}g(x) + \int_0^1 K(x,t)\,\alpha_n(t)\,\mathrm{d}t + C_n(x)\,. \tag{2.5}$$

Employing relations (2.4) in (2.5) yields

$$C_n(x) = -[\beta_{n+1}g(x) + \beta_n g_1(x) + \beta_{n-1}g_2(x) + \ldots + g_{n+1}(x)]\,, \tag{2.6}$$

where

$$g_k(x) = \int_0^1 K(x,t)\,g_{k-1}(t)\,\mathrm{d}t\,, \qquad 1 \leqslant k \leqslant n+1, \qquad g_0(x) = g(x)\,.$$

One way of determining the β_k is to minimize $C_n(x)$ in the L^2 norm, i.e. choose β_k such that

$$H(\beta_1, \ldots \beta_n) = \int_0^1 C_n^2(x)\,\mathrm{d}x = \min\,. \tag{2.7}$$

This is motivated by the observation that if K is separable, the approximation will be exact if n is large enough (see theorem 1).

The minimization in (2.7) is effected by solving the equations

$$\frac{\partial H(\beta_1, \ldots \beta_{n+1})}{\partial \beta_i} = 0\,, \qquad i = 1, \ldots n+1\,. \tag{2.8}$$

Equation (2.8) can be expressed as the matrix equation

$$(k_{i,j})(\beta_j) = -(k_{i,0})\,, \tag{2.9}$$

where

$$k_{i,j} = \int_0^1 g_{n-i+1}(x)\,g_{n-j+1}(x)\,\mathrm{d}x\,, \qquad i,j = 1, 2, \ldots n+1\,.$$

Notice that $k_{i,j} = k_{j,i}$ so that the coefficient matrix in (2.9) is symmetric, and that increasing the order of approximation results in bordering the matrix of coefficients in (2.9), thus, information obtained in a given approximation can be used to advantage in the computation of the next higher order approximation.

If the kernel in (2.1) is separable,

$$K(x, t) = \sum_{i=1}^{m} H_i(x)\, J_i(t)\,, \tag{2.10}$$

and we set

$$u_i = \int_0^1 J_i(t)\, g(t)\, \mathrm{d}t\,,$$

$$v_{ij} = \int_0^1 J_i(t)\, H_j(t)\, \mathrm{d}t\,,$$

and

$$\boldsymbol{V} = (v_{i,j})\,, \qquad \boldsymbol{U} = (u_1, \dots u_m)^{\mathrm{t}}\,, \qquad i, j = 1, 2, \dots m\,, \tag{2.11}$$

then the k_{ij} in (2.9) can be written as

$$k_{i,j} = (\boldsymbol{V}^{n-i}\boldsymbol{U})^{\mathrm{t}}\, \boldsymbol{H}(\boldsymbol{V}^{n-j}\boldsymbol{U})\,, \tag{2.12}$$

where

$$\boldsymbol{H} = (h_{i,j})\,, \qquad h_{i,j} = \int_0^1 H_i(t)\, H_j(t)\, \mathrm{d}t\,.$$

Thus (2.9) can be written as

$$\boldsymbol{U}^{\mathrm{t}}(\boldsymbol{V}^{-1})^{\mathrm{t}} \left\{ (\boldsymbol{V}^{n-i+1})^{\mathrm{t}}\, \boldsymbol{H} \left[\sum_{j=0}^{n+1} \beta_j\, \boldsymbol{V}^{n-j+1} \right] \right\} \boldsymbol{V}^{-1}\boldsymbol{U} = \boldsymbol{o}\,, \qquad i = 0, 1, \dots n+1\,. \tag{2.13}$$

Since $K(x, t)$ is separable, it is easily verified from the theory of the separable case that V satisfies an equation of the form

$$c_0\boldsymbol{I} + c_1\boldsymbol{V} + \dots + c_m\boldsymbol{V}^m = \boldsymbol{O}\,, \tag{2.14}$$

so that, if $n+1 = m$ in (2.13), then

$$\beta_j = c_{m-j}/c_0\,, \qquad j = 1, \dots n+1\,,$$

is a solution of (2.13). Since the solution is unique, the approximation is exact in that it coincides with the analytical solution. Thus,

THEOREM 1: If the kernel of (2.1) is separable, the approximation (2.2) with $n = m-1$ is exact.

Since compact kernels can be approximated uniformly by separable kernels, we can state

THEOREM 2: If the kernel in (2.1) is compact, the sequence of approximations (2.2) converge uniformly to the solution of (2.1).

The above method depends on the evaluation of integrals to determine α_k, g_k and $k_{i,j}$. Of course the integrals cannot usually be evaluated analytically, and numerical methods must be used. Also, the equations for the determination of the β_k are likely to be ill-conditioned unless n is not too large. In fact, the practicality of the method for large values of n will depend greatly on the character of K and of the choice of the perturbation term R_n.

The approximations developed here can be related to the following sequence of successive approximations to the solution of (1.4),

$$F_0 = g \quad \text{and} \quad F_{n+1} = g + \lambda \int_0^1 K(x, t)\, F_n(t)\, \mathrm{d}t\,, \tag{2.15}$$

by observing that

$$\frac{A_n}{B_{n+1}} = \sum_{j=0}^{n} h_{n,j}\, F_{n-j}\,, \qquad h_{n,j} = \frac{\beta_j \lambda^j}{B_{n+1}}\,, \qquad \sum_{j=0}^{n} h_{n,j} = 1 - \frac{\beta_{n+1}\lambda^{n+1}}{B_{n+1}}\,, \tag{2.16}$$

$$-\lambda^{n+1} C(x) = \sum_{j=0}^{n} \lambda^i \beta_j (F_{n-j+1} - F_{n-j}) + g\,\beta_{n+1}\lambda^{n+1}\,.$$

However, it does not appear that our approximations can be directly related to approximations obtained by (2.15) in that the determination of the β_j is not facilitated by properties of the F_j.

3. Interpolatory approximations

The philosophy of the τ-method dictates that the most satisfactory choice of an appended term would be a polynomial, $R_n(\lambda)$, which has its zeros in a λ-region where we desire to compute the solution of the integral equation. Indeed, numerical evidence indicates that the choice of such a term yields an approximation with smaller maximum error over $0 \leqslant x \leqslant 1$ than the choice of λ^{n+1}, but there are a number of difficulties involved, not the least of these being that the equations for the α_j, β_j can no longer be solved explicitly. We indicate how this difficulty can be partly overcome.

The equation is now

$$f_n(x) = g(x) + \lambda \int_0^1 K(x, t)\, f_n(t)\, \mathrm{d}t + \frac{C_n(x)\, R_{n+1}(\lambda)}{B_{n+1}(\lambda)}\,, \tag{3.1}$$

where as before

$$\left.\begin{aligned} f_n(\lambda) &= \frac{A_n(\lambda, x)}{B_{n+1}(\lambda)} \\ A_n(\lambda, x) &= \alpha_0 + \alpha_1\lambda + \alpha_2\lambda_2 + \ldots + \alpha_n\lambda^n \\ B_{n+1}(\lambda) &= 1 + \beta_1\lambda + \beta_2\lambda^2 + \ldots + \beta_{n+1}\lambda^{n+1} \end{aligned}\right\}\,, \tag{3.2}$$

and the α_j depend on x, but not the β_j.

Let

$$R_{n+1}(\lambda) = \mu_0 + \mu_1 \lambda + \dots + \mu_{n+1} \lambda^{n+1} . \tag{3.3}$$

We find that

$$\alpha_k = \beta_k g(x) + \int_0^1 K(x, t)\alpha_{k-1}(t)\, \mathrm{d}t + C_n(x)\mu_k , \qquad k = 0, 1, 2, \dots n+1 , \qquad \alpha_{k-1} = \alpha_{n+1} = 0 . \tag{3.4}$$

Solving the last equation for C_n gives

$$\left.\begin{aligned} C_n(x) &= -L(\alpha_n)/\mu_{n+1} \\ L(u) &= \int_0^1 K(x, t)\, u(t)\, \mathrm{d}t . \end{aligned}\right\} ; \tag{3.5}$$

so the equations (3.4) become

$$\alpha_k = \beta_k\, g(x) + L\left[\alpha_{k-1} - \frac{\alpha_n\, \mu_k}{\mu_{n+1}}\right], \qquad k = 0, 1, 2, \dots n+1 . \tag{3.6}$$

Because of the presence of α_n in each equation, these equations can no longer be solved by recursion. There is, in general, no way of overcoming this complication. However, the following iterative procedure has yielded very effective rational approximations in a number of examples. Suppose the β_k, α_k and $C_n(x)$ for $R_{n+1}(\lambda) = \lambda^{n+1}$ are available. They can be substituted into the *right hand* side of (3.6) to yield new α_k. One hopes this would yield an approximation

$$f(x) \sim \sum_{k=0}^{n} \alpha_k^* \lambda^k \Big/ \sum_{k=0}^{n+1} \beta_k\, \lambda^k , \tag{3.7}$$

whose error curve is more nearly level than the curve corresponding to the approximation $\Sigma\, \alpha_k\, \lambda^k / \Sigma\, \beta_k\, \lambda^k$. In several examples that we have tried, this is the case.

The question of whether the equations can be solved in closed form is open.

4. Error bounds

For moderate values of λ, a theoretical error bound for the approximation f_n defined by (3.1) can be easily determined. Subtracting (3.1) from (2.1) gives

$$E_n(x) = \lambda \int_0^1 K(x, t)\, E_n(t)\, \mathrm{d}t - \frac{C_n(x)\, R_{n+1}(\lambda)}{B_{n+1}(\lambda)} . \tag{4.1}$$

Let

$$\|E_n\| = \sup_{0 \leqslant x \leqslant 1} |E_n(x)| . \tag{4.2}$$

Then

$$|E_n(x)| \leqslant |\lambda| \, \|E_n\| \int_0^1 |K(x, t)| \, dt + \frac{|C_n(x)| \, |R_{n+1}(\lambda)|}{|B_{n+1}(\lambda)|} \leqslant |\lambda| \, \|E_n\| \, k + C_n \frac{|R_{n+1}(\lambda)|}{|B_{n+1}(\lambda)|} .$$

$$k = \sup_{0 \leqslant x \leqslant 1} \int_0^1 |K(x, t)| \, dt , \qquad C_n = \sup_{0 \leqslant x \leqslant 1} |C_n(x)| .$$

Thus, taking the sup of the left hand side yields

$$\|E_n\| \leqslant \frac{C_n |R_{n+1}(\lambda)|}{|B_{n+1}(\lambda)|(1 - |\lambda| k)} = v_n , \qquad k < 1/|\lambda| . \tag{4.3}$$

5. Examples

For the data in this section, we treat the problem

$$f(x) = x + \lambda \int_0^1 K(x, t) f(t) \, dt ,$$

$$K(x, t) = \begin{cases} x(t-1), & 0 \leqslant x \leqslant t \\ t(x-1), & t \leqslant x \leqslant 1 \end{cases} \tag{5.1}$$

(see [4, p. 121]). The eigenvalues are $\lambda_k = -k^2\pi^2$, $k = 1, 2, ...$ For $\lambda \neq \lambda_k$, the solution f is given by

$$f(x) = x + 2\lambda \sum_{k=0}^{\infty} (-1)^k \frac{\sin(\pi k x)}{\pi k(\lambda + k^2\pi^2)} . \tag{5.2}$$

Example 1: Here we compare $\|E_2\|$ and v_n (see (4.2), (4.3)) for $R_2(\lambda) = \lambda^2$

λ	$\|E_2\|$	v_n
−2	3.2×10^{-5}	6.7×10^{-5}
−4	6.9×10^{-4}	11.4×10^{-4}

Example 2: One of the more efficient analytical approximations to linear functional equations is the *moment method* [7, p. 258]. For Fredholm integral equations these approximations are quotients of polynomials (in λ) of the same degree. In order to compare our noninterpolatory approximations (2.3) with those of the moment method, we adjusted our approximations so that they also are quotients of polynomials in λ of the same degree. The linearly independent set of

functions on [0, 1] employed in the moment method were chosen to be positive powers of the variable x. Thus, in both approximations our noninterpolatory approximations and those developed by the moment method consist of quotients of polynomials in λ in which the numerator coefficients involve powers of x. As a matter of fact, the numerators of both approximations are of the same degree as a polynomial in x.

In the tables below, n is the order of the approximations, ϵ_n denotes the maximum absolute error for the adjusted approximations (2.3), and e_n is the maximum absolute error incurred by the approximations derived by the moment method. The number in parentheses is the power of 10 by which the number it appears next to must be multiplied.

$\lambda = -2$			$\lambda = -4$			$\lambda = -6$		
n	ϵ_n	e_n	n	ϵ_n	e_n	n	ϵ_n	e_n
2	.34(−4)	.21(−1)	2	.56(−3)	.41(−1)	2	.31(−2)	.63(−1)
3	.27(−6)	.20(−2)	3	.54(−5)	.10(−1)	3	.45(−4)	.36(−1)
4	.46(−7)	.12(−3)	4	.93(−7)	.50(−3)	4	.65(−6)	.15(−2)
5	.45(−7)	.60(−4)	5	.92(−7)	.18(−3)	5	.18(−6)	.30(−2)

The accuracy of the approximations (2.3) is remarkable as compared to that of the approximations derived by the moment method.

We have used the iterative procedure discussed in section 3 to improve the above approximations, taking for $R_n(\lambda)$ the Laguerre polynomial $L_n(\lambda)$. The result was an increase in accuracy of about one decimal place.

6. Conclusions

On the basis of our exploratory numerics, the application of the τ-method in its present formulation to Fredholm integral equations of the second kind seems well worth pursuing. Our choice of the perturbation term and the expression to be minimized made the problem tractable and leads to effective approximations for small values of n. Other choices of the perturbation term and methods of determining the coefficients should be investigated. The method is limited by the difficulty of computing the iterated kernels and solving the resulting equations. However, the lower order approximants furnish good values for starting an acceptable iterative procedure and can give information regarding eigenvalues in that the zeros of the denominator polynomial approximate the eigenvalues. Indeed, we recommend that the approximations be used in this way until a more thorough analysis is available.

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A Transformation for the Economization of Some Toeplitz Summation Methods

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A Toeplitz array is a triangular array of numbers

$$U=\left\{\mu_{n,k}\middle|0\leqq k\leqq n; n=0,1,2,\ldots,; \sum_{k=0}^{n}\mu_{n,k}=1\right\}.$$

Given a sequence $\{s_n\}$ one may define the transformation

$$U: s_n\to\bar{s}_n, \bar{s}_n=\sum_{k=0}^{n}\mu_{n,k}s_k, n=0,1,2,\ldots.$$

In what follows we assume the given sequence is defined as the n-th partial sum of some series

$$s_n=\sum_{j=0}^{n}a_j$$

so our discussion will be oriented toward improving convergence of series. If

$$\lim_{n\to\infty}s_n=s=\sum_{j=0}^{\infty}a_j$$

we will write

$$r_n = s - s_n,$$

$$\bar{r}_n = \sum_{k=0}^{n} \mu_{n,k} r_k$$

for the error and transformed error sequences, respectively. If $\bar{r}_n = o(1)$ for a convergent sequences (or for a class of convergent sequences) U is said to b regular (or regular for that class of sequences). For a discussion of Toeplit transformations, see [1], [2], [3].

$$P_n(t) = \sum_{k=0}^{n} \mu_{n,k} t^k$$

is called the n-th characteristic polynomial of U. The accompanying table give some important regular methods and their characteristic polynomials.

In the case where Σa_i is a series whose terms are monotone decreasing regular positive methods ($\mu_{n,k} \geqq 0$) are generally inefficient. The reason is no hard to discover for, in such cases, s_n is as close to s as any of $s_0, s_1, \ldots, s_{n-1}$, ye $\bar{s}_n$ is a positive average of $s_0, s_1, \ldots, s_n$, so s_n is as close to s as $\bar{s}_n$. Positiv methods do have the advantage, however, of being numerically stable.

TABLE I

Name	$\mu_{n,k}$	$P_n(t)$
Identity	$0, 0 \leqq k \leqq n-1$ $1, k = n$	t^n
Cesaro	$\frac{1}{n+1}$	$\frac{1-t^{n+1}}{(n+1)(1-t}$
Binomial	$\binom{n}{k} \lambda^{n-k}/(\lambda+1)^n, \quad \lambda > 0$	$\left(\frac{\lambda+t}{\lambda+1}\right)$
Chebyshev	$\frac{2\binom{n}{k}(n)_k \lambda^k}{\left(\frac{1}{2}\right)_k (\sigma^{2n} + \sigma^{-2n})}, \quad \lambda > 0$ $\sigma = \lambda^{1/2} + \sqrt{\lambda+1}$	$\frac{T_n(2\lambda t+1}{T_n(2\lambda+1}$

Recently, the author has discussed classes of non-regular methods ([4], [5], 6], [7]). The methods work well on summing series whose terms are nonotone decreasing but they are numerically unstable since the $\mu_{n,k}$ become arge with n and alternate in sign.

Our idea is to "correct" $\bar{s}_n$ by adding a term $c_n\,\bar{a}_n$ where c_n is independent of he sequence s_n and depends only on the method U. When the a_n are difficult to ompute, as they often are (in practical situations, they may involve omplicated products of higher mathematical functions) the method is useful n gaining additional accuracy at little computational cost. Also, the method nay be repeated on the new sequence so obtained.

When U is regular the method is numerically stable. When U is not regular nd numerically unstable the method still yields additional figures.

Let M denote the class of sequences

$$M=\left\{u_n \,\middle|\, u_n=\int t^n\psi,\ \frac{\psi}{1-t}\in L^2[0,1]\right\}.$$

)efine

$$\tau_r\equiv\tau_r(n,U)=\int P_n^2(1-t)^r,\quad r=0,1,2,\ldots.$$

All integrals in this paper are with respect to t and between limits 0 and 1 nless indicated otherwise. All order symbols are with respect to n.)

In the derivation, we assume $a_n\in M$. If $\psi\geqq 0$ then a_n is monotone. In fact M ontains many of the monotone sequences of interest in practical applications.

We find

$$s_n=\int\frac{1-t^{n+1}}{1-t}\psi,\quad s=\int\frac{\psi}{1-t},$$

$$r_n=\int\frac{t^{n+1}}{1-t}\psi,\ n=0,1,2,\ldots.$$

Jow we wish to determine c in the equation

$$\begin{aligned}\hat{s}_n&=\bar{s}_n+c\bar{a}_n\\&=\sum_{k=0}^{n}\mu_{n,k}s_k+c\sum_{k=0}^{n}\mu_{n,k}a_k,\end{aligned}$$

o that

$$\hat{r}_n=s-\hat{s}_n=\bar{r}_n-c\bar{a}_n$$

is, in some sense, minimized. The constant c will depend on n and will be chacteristic of the method U.

An easy calculation gives

$$\hat{r}_n = \int \frac{[t-c(1-t)]}{(1-t)} \psi P_n.$$

By Schwarz's inequality

$$|\hat{r}_n| \leqq \left\| \frac{\psi}{1-t} \right\|_2 [g(c)]^{1/2}.$$

It is natural to pick c to minimize

$$g(c) = \int [t-c(1-t)]^2 P_n^2.$$

It is easily shown

$$c \equiv c_n \equiv c_n(U) = -1 + \frac{\tau_1}{\tau_2}$$

produces this minimum. Also

$$g(c_n) \equiv \tau_0 - \frac{\tau_1^2}{\tau_2}.$$

Since c_n minimizes $g(c)$, putting $c = -1$ in $g(c)$ shows

$$g(c_n) = \int P_n^2 [t - c_n(1-t)]^2 \leqq \int P_n^2$$

so

$$|\hat{r}_n| \leqq \left\| \frac{\psi}{1-t} \right\|_2 \|P_n\|_2.$$

Let U be regular. Since $\overline{t^n} = P_n(t)$,

$$\begin{aligned} \lim_{n\to\infty} P_n(t) &= 0, 0 \leqq t < 1 \\ &= 1, t = 1. \end{aligned}$$

Also,

$$|P_n(t)| \leq \sum_{k=0}^{n} \mu_{n,k} < M, \quad t \in [0,1],$$

by a known result for regular methods (see [1]). The Lebesgue convergence theorem shows that

$$\lim_{n\to\infty} \|P_n\|_2 = 0$$

and we have:

THEOREM 1 *If U is regular, the method defined by $\hat{s}_n$ is regular when $a_n \in M$.*

We can write

$$\hat{s}_n = \bar{s}_n + c_n \bar{a}_n = \sum_{k=0}^{n} [\mu_{n,k} + c_n(\mu_{n,k} - \mu_{n,k+1})] s_k$$

where $\mu_{n,n+1} = 0$. An application of the Toeplitz limit theorem [2, p. 43] gives:

THEOREM 2 *Let U be regular. Then the method given by $\hat{s}_n$ is regular if and only if both*

a) $c_n(\mu_{n,k} - \mu_{n,k+1}) = o(1)$, $k = -1, 0, 1, 2, \ldots, n$, $(\mu_{n,-1} = 0)$,

b) $\sum_{k=0}^{n} |\mu_{n,k} + c_n(\mu_{n,k} - \mu_{n,k+1})| = 0(1)$.

In Table II we have given an example for the binomial weights and in Table III we have tabulated c_n for the methods of Table I and given asymptotic estimates for large n. The estimate of c_n for the Chebyshev weights is based on

TABLE II

$$s_n = \sum_{k=0}^{n} \frac{1}{(k+1)(k+2)}, \quad \mu_{n,k} = \binom{n}{k} 2^{-n}, s = 1$$

n	$s - \bar{s}_n$	$s - \hat{s}_n$
5	2.4×10^{-1}	3.6×10^{-2}
10	1.5×10^{-1}	8.0×10^{-3}
30	6.0×10^{-2}	1.0×10^{-3}
50	3.8×10^{-2}	3.8×10^{-4}

TABLE III

Name	$K(t)$	c_n	Asymptotic c_n
Identity	t	$(n+1/2)$	$n(1+o(1))$
Cesaro	1	$\frac{(n+2)(2n+3)}{2(n+1)^2}[2h_{n+2}-h_{2n+3}-1]$ $h_n = 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}, n \geqq 1, h_0 = 1$	$\ln n(1+o(1))$
Binomial	$\frac{\lambda+t}{\lambda+1}$	$\frac{(2n+1-2\lambda)+\rho^{2n+2}(2n+2\lambda+3)}{(2\lambda+2)-\frac{2\rho^{2n+2}}{\lambda}(\lambda^2+(2n+3)(n+\lambda+1))}$ $\rho = [\lambda/(\lambda+1)]$	$\frac{2n+1-2\lambda}{2\lambda+2}+o(n\rho^{2n})$
Chebyshev	$\left[\frac{\sqrt{\lambda t}+\sqrt{\lambda t+1}}{\sqrt{\lambda}+\sqrt{\lambda+1}}\right]^2$	$\frac{1+{}_3F_2\left(\begin{matrix}-2n, 2n, 2\\ \frac{1}{2}, 4\end{matrix}\middle\vert -\lambda\right)}{2\left(1+{}_3F_2\left(\begin{matrix}-2n, 2n, 1\\ \frac{1}{2}, 4\end{matrix}\middle\vert -\lambda\right)\right)}$	$\frac{n\sqrt{\lambda}}{2\sqrt{\lambda+1}}(1+o(1))$

he work of Fields and Luke [8]. One consequence of Theorem 2 is that, for the binomial weights with $\lambda=1$, $\hat{s}_n$ is not regular. Let

$$M_n=\sum_{k=0}^{n}\left|\mu_{n,k}-\mu_{n,k+1}\right|.$$

Then, by symmetry,

$$M_{2n}=2(\mu_{2n,n}-\mu_{2n,0})$$

so

$$c_{2n}M_{2n}\sim 2\sqrt{\frac{n}{\pi}},\quad n\to\infty.$$

Clearly, condition (b) is violated.

A look at the Table III tempts one to conjecture that $c_n\to\infty$ as $n\to\infty$. This is not generally true, even for positive regular methods. Let U be the regular method given by

$$P_n(t)=[(n+1)t^n+1]/(n+2).$$

A straightforward computation shows that here

$$c_n=\tfrac{5}{4}+0(n^{-1}).$$

We do have the result:

THEOREM 3 *Let U be positive and $nP_n(x)=o(1)$ for each $x\in[0,1)$. Then*

$$\frac{c_n}{1+c_n}=1-\delta_n$$

where δ_n is a positive null sequence.

Proof Let

$$v_n=1-\frac{c_n}{1+c_n},$$

$$P_n^2(t)=\sum_{k=0}^{2n}a_kt^k.$$

We have

$$0 \leqq v_n = \frac{\tau_2}{\tau_1} = \left(\int_0^\eta P_n^2(1-t)^2 + \int_\eta^1 P_n^2(1-t)^2\right)/\tau_1$$

$$\leqq \left[\int_0^\eta P_n^2(1-t)^2 + (1-\eta)\tau_1\right]/\tau_1 = (1-\eta) + \frac{\int_0^\eta P_n^2(1-t)^2}{\tau_1}.$$

Now

$$\tau_1 = \sum_{k=0}^{2n} a_k/(k+1)(k+2)$$

so

$$0 \leqq v_n \leqq (1-\eta) + \frac{\displaystyle\sum_{k=0}^{2n} \frac{a_k \eta^{k+1}}{k+1}}{\displaystyle\sum_{k=0}^{2n} \frac{a_k}{(k+1)(k+2)}}$$

$$\leqq (1-\eta) + \frac{\eta P_n^2(\eta)}{\dfrac{P_n^2(1)}{(2n+1)(2n+2)}}$$

$$= (1-\eta) + (2n+1)(2n+2)\eta P_n^2(\eta).$$

Now given $\varepsilon > 0$ pick $1-\eta < \varepsilon/2$ and n_0 *such that*

$$(2n+1)(2n+2)\eta P_n^2(\eta) < \frac{\varepsilon}{2}$$

for $n > n_0$. Then

$$0 \leqq 1 - \frac{c_n}{1+c_n} < \varepsilon, \quad n > n_0$$

and the theorem is proved.

In fact, since

$$c_n = \frac{\sum a_k/(k+2)}{\sum a_k/(k+1)(k+2)}$$

we have, for positive methods,

$$1 \leqq c_n \leqq 2n+1.$$

An estimate for τ_r is useful. The following theorem often provides the desired information.

THEOREM 4 *Let*

$$n\left(\frac{|P_n(t)|^{1/n}}{K(t)}-1\right)$$

be uniformly bounded as $n \to \infty$ for $t \in [0, 1]$ for some positive integrable $K(t)$. Further for some $\eta > 0$ let

a) $K(t) \in C^1[1-\eta, 1)$.
b) $K(t) \leqq 1-\varepsilon, t \in [0, 1-\eta]$ *for some* $\varepsilon > 0$.
c) $K'(t) \sim a(1-t)^m, a \neq 0, m > -1, t \to 1$.

Then there are positive constants α_1, α_2 and a positive integer n_0 *such that*

$$\alpha_1 < \tau_r n^{\frac{r+1}{m+1}} < \alpha_2, \quad n > n_0.$$

Proof We have

$$\frac{B}{n} < \frac{|P_n|^{1/n}}{K} - 1 < \frac{A}{n}, \quad n > n_0$$

or

$$\left(1+\frac{B}{n}\right)^{2n} K^{2n} < P_n^2 < \left(1+\frac{A}{n}\right)^{2n} K^{2n}, \quad n > n_0,$$

n_0 being chosen such that the terms

$$1+\frac{B}{n}, \; 1+\frac{A}{n},$$

are positive for $n > n_0$.

An upper bound for τ_r is

$$\left(1+\frac{A}{n}\right)^{2n} \int K^{2n}(1-t)^r = \frac{e^{2A}}{m+1} \Gamma\left(\frac{r+1}{m+1}\right)\left(\frac{m+1}{2an}\right)^{\frac{r+1}{m+1}} \quad (1+o(1)).$$

the order estimate on the right following from a result in Erdélyi [9, p. 37]. A lower bound is calculated in the same way.

If the stronger condition

$$n^{1+\sigma}\left(\frac{|P_n|^{1/n}}{K}-1\right) \quad \text{uniformly bounded}$$

as $n\to\infty$ in $[0, 1]$ for some $\sigma>0$ we find

$$\tau_r=\frac{\Gamma\left(\frac{r+1}{m+1}\right)}{(m+1)}\left(\frac{m+1}{2an}\right)^{\frac{r+1}{m+1}} \quad (1+o(1)).$$

The conditions of Theorem 4 are satisfied for all the methods given in Table III except for the Cesaro method. For the identity, binomial and Chebyshev methods $m=0$ and $a=1$, $(\lambda+1)^{-1}$ and $[\lambda/(\lambda+1)]^{1/2}$ respectively. For the Cesaro method we have

$$P_n(t)=\frac{1+t+t^2+\ldots+t^n}{n+1}$$

so on $[0, 1]$

$$\frac{1}{n+1}\leqq P_n(t)\leqq 1$$

(both bounds being attained on $[0, 1]$). Thus

$$0\leqq 1-P_n^{1/n}\leqq 1-(n+1)^{-1/n}$$

and

$$0\leqq n(1-P_n^{1/n})\leqq n\left(1-\exp\left[\frac{-\ln n-\ln(1+1/n)}{n}\right]\right)$$

$$=n\left(1-\exp\left[\frac{-\ln n}{n}+0(n^{-2})\right]\right)$$

$$=\ln n+0\left(\frac{(\ln n)^2}{n}\right)$$

so the given quantity is not uniformly bounded. (Note K must be the pointwise limit of $|P_n|^{1/n}$ if the latter exists.) For the Cesaro method we have, in fact,

$$\tau_0 = \frac{2}{(n+1)}[\psi(n+1)+\psi(2n+2)] = \frac{2\ln 2}{n^2}[1+o(n^{-1})]$$

$$\tau_1 = \frac{1}{(n+1)^2}[2\psi(n+2)-\psi(2n+3)] = \frac{\ln n}{n^2}+0(n^{-2})$$

$$\tau_r = \frac{1}{n^2(r-1)}+0(n^{-r-1}), \quad r>1.$$

The more strongly ψ vanishes at 1, the more efficient the method is. Let

$$\psi = (1-t)^{r+1}h, \; h \in L^2[0,1].$$

Then

$$|\hat{r}_n| \leqq \|h\|_2 v(c_n)^{1/2},$$

$$v(c_n) = \int P_n^2(1-t)^{2s}[t-c(1-t)]^2$$

$$= \tau_{2s} - 2\frac{\tau_1}{\tau_2}\tau_{2s+1} + \frac{\tau_1^2}{\tau_2^2}\tau_{2s+2}.$$

If, say, $m=1$, we find

$$v(c_n)^{1/2} = \frac{A}{n^{s+1/2}}[1+o(1)].$$

Our experience shows this bound is too conservative—the improvement $\hat{s}_n$ offers is much greater than indicated.

An open question is: does there exist a regular method U which minimizes

$$\int P_n^2[t-c_n(U)(1-t)^2]\,?$$

The iteration of the method is straightforward. Just let

$$\hat{a}_n = \hat{s}_n - \hat{s}_{n-1}, \quad n=1,2,\ldots,$$
$$\hat{a}_0 = \hat{s}_0$$

and define

$$\hat{\hat{s}} = \bar{\bar{s}}_n + c_n\bar{\bar{a}}_n.$$

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Differential-Difference Properties of Hypergeometric Polynomials*

By Jet Wimp

Abstract. We develop differential-difference properties of a class of hypergeometric polynomials which are a generalization of the Jacobi polynomials. The formulas are analogous to known formulas for the classical orthogonal polynomials.

I. Introduction. In this paper, we derive a differential-difference equation satisfied by the hypergeometric polynomials

$$(1)\qquad P_n(x) = {}_{p+2}F_q\left(\begin{matrix} -n, n+\lambda, a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{matrix}\,\middle|\, x\right), \qquad n = 0, 1, 2, \cdots.$$

Throughout, we employ the shorthand notation

$$(2)\qquad (a_p + n) = \prod_{j=1}^{p} (a_j + n), \text{ etc.},$$

see [1]. In general, where any variable is subscripted by a p or q, it is to be understood that the shorthand notation has been invoked.

II. Results.

THEOREM. *Let*

(i) $\lambda \neq 1, 2, \cdots$;

(ii) *none of the quantities* $b_j, \lambda, \lambda + 1 - b_j$ *be negative integers or zero*, $j = 1, 2, \cdots, q$;

(iii) *no* $b_j =$ *any* a_h, $h = 1, 2, \cdots, p$; $j = 1, 2, \cdots, q$. *Then the polynomials* $P_n(x)$ *satisfy the differential-difference equation*

$$(3)\qquad (\epsilon x - \delta x^2)\frac{dP_n(x)}{dx} = \sum_{\nu=0}^{\sigma} (A_\nu + xB_\nu)P_{n-\nu}(x),$$

where

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$$(4)\quad \delta = \begin{cases} 1, & p+1 \geqslant q, \\ 0, & p+1 < q, \end{cases} \qquad \epsilon = \begin{cases} 0, & p+1 > q, \\ 1, & p+1 \leqslant q, \end{cases} \qquad \sigma = \max\{p+1, q\},$$

and no such equation of lower order $\sigma' < \sigma$ exists. The A_ν's and B_ν's are unique and

$$(5)\quad A_\nu = \begin{cases} (-n)_\nu [(1-n-\lambda)_\nu]^{-1} (2\nu - 2n - \lambda) \\ \qquad \cdot \left\{ (-)^{\nu+1}\epsilon + \dfrac{(-)^{\sigma+1}}{\nu!} \displaystyle\sum_{s=0}^{\nu} \dfrac{(-\nu)_s (n-s)(b_q + n - s - 1)}{(\nu + s - 2n - \lambda)_{\sigma+1-\nu}} \right\}, & \nu > 0; \\ n \left\{ \epsilon - \dfrac{(b_q + n - 1)}{(2n + \lambda - \sigma)_\sigma} \right\}, & \nu = 0; \end{cases}$$

$$(6)\quad B_\nu = \begin{cases} (-n)_\nu [(1-n-\lambda)_\nu]^{-1} (2\nu - 2n - \lambda) \\ \qquad \cdot \left\{ (-)^{\nu}\delta + \dfrac{(-)^{\sigma+1}}{\Gamma(\nu)} \displaystyle\sum_{s=0}^{\nu-1} \dfrac{(1-\nu)_s (a_p + n - s - 1)}{(\nu + s + 1 - \lambda - 2n)_{\sigma-\nu}} \right\}, & \nu > 0, \\ -\delta n, \quad \nu = 0. \end{cases}$$

Proof. By equating coefficients of x^{k+1} in (3) we find

$$(7)\quad \begin{aligned} &(k+1)\{\epsilon(k-n)(a_p+k)\beta_{-1}(k) - \delta k(b_q+k)\beta_0(k)\} \\ &\equiv (a_p+k)\sum_{\nu=0}^{\sigma} C_\nu \alpha_{\nu+1}(k)\beta_{\nu-1}(k) + (k+1)(b_q+k)\sum_{\nu=0}^{\sigma} D_\nu \alpha_\nu(k)\beta_\nu(k), \end{aligned}$$

where

$$(8)\quad \begin{bmatrix} C_\nu \\ D_\nu \end{bmatrix} = \frac{(-)^\nu (1-n-\lambda)_\nu}{(-n)_\nu} \begin{bmatrix} A_\nu \\ B_\nu \end{bmatrix}, \qquad \alpha_\nu(k) = (k-n)_\nu, \quad \beta_\nu(k) = (n+\lambda+k-\sigma)_{\sigma-\nu}.$$

The above can be considered an identity between polynomials in the (generally complex-valued) variable k. If $p+1 > q$, (7) requires that two polynomials of degree $p + \sigma + 2$ be identical; this condition furnishes $p + \sigma + 3$ equations in $2\sigma + 2$ unknowns, so that we must have $\sigma \leqslant p + 1$. If $p + 1 = q$, we similarly find $\sigma \leqslant p + 1$, while, if $p + 1 < q$, we find that $\sigma \leqslant q$. Thus

$$(9)\quad \sigma \leqslant \max\{p+1, q\}.$$

Now, if we assume equality above, the A_ν and B_ν (if they exist) are unique. Suppose there is another such recurrence relation with coefficients A_ν^* and B_ν^*. Subtracting these two, we have

$$(10)\qquad 0=\sum_{\nu=0}^{\sigma}[(A_\nu-A_\nu^*)+x(B_\nu-B_\nu^*)]P_{n-\nu}(x),$$

but this is impossible, under the hypotheses (ii) and (iii), since the author has shown that in this case any linear difference equation satisfied by $P_n(x)$ must be of order $\sigma+1$, at least, see [1].

Now, if $q=p+1$, (7) holds if and only if

$$(11)\qquad (k+1)[\beta_0(k+1)\epsilon-(b_q+k)]\equiv\sum_{\nu=0}^{\sigma}C_\nu\alpha_\nu(k+1)\beta_\nu(k+1),$$

$$(12)\qquad (n+\lambda+k-\sigma)[-\delta k\beta_1(k+1)+(k-n)(a_p+k)]\equiv\sum_{\nu=0}^{\sigma}D_\nu\alpha_\nu(k)\beta_\nu(k).$$

(Note that a suitable linear combination of (11) and (12) gives (7), i.e., multiply (11) by $(k-n)(n+\lambda+k-\sigma)(a_p+k)$ and (12) by $(k+1)(b_q+k)$ and add.) To establish (11) for $p+1=q$, we observe that it represents an identity between two polynomials in k, each of degree $q+2$ and each having two identical factors. It only remains to show that (11) holds for $q+1$ distinct values of k. Assume that all the quantities $-1, -b_j, j=1,2,\cdots,q$, are distinct and let k have these values in (7). The result is (11) evaluated at these values.

Similarly to show (12) for $p+1=q$, we need only prove that it holds for the $p+2$ values (assumed distinct) $n, \sigma-n-\lambda, -a_j, j=1,2,\cdots,p$. This is true, since (7) and (12) for these values are the same.

(The requirement that the values of k chosen above be distinct may be relaxed by continuity.)

Now, replace x by $x/a_j, j=p'+1,p'+2,\cdots,q-1$ in (3), where $p'<q-1$. This shows that

$$(13)\qquad \epsilon x\frac{dP_n'(x)}{dx}=\sum_{\nu=0}^{\sigma}(C_\nu+xD_\nu')P_{n-\nu}(x)(-)^\nu(1-n-\lambda)_\nu/(-n)_\nu,$$

where $P_n'(x)$ is $P_n(x)$ with p replaced by p' and

$$(14)\qquad D_\nu'=\lim_{a_u\to\infty}\lim_{a_{u+1}\to\infty}\cdots\lim_{a_v\to\infty}[D_\nu/a_ua_{u+1}\cdots a_v],\qquad u=p'+1,\ v=q-1.$$

The same limit process applied to (12) yields the following equation for the determination of D_ν':

$$(15)\qquad (k-n)(n+\lambda+k-\sigma)(a_{p'}+k)=\sum_{\nu=0}^{\sigma}D_\nu'\alpha_\nu(k)\beta_\nu(k).$$

The equation for C_ν in this case is (11) as it stands. But (11) and (15) together are (11) and (12), respectively, written for $p+1<q$.

Similarly, replacing x by xb_j, $j = q' + 1, q' + 2, \cdots, p + 1, q' \leqslant p$, and letting $b_j \to \infty$ in (3) gives

$$(16) \qquad -x^2 \frac{dP''_n(x)}{dx} = \sum_{\nu=0}^{\sigma} (C'_\nu + xD_\nu)P''_{n-\nu}(x)(-)^\nu (1 - n - \lambda)_\nu/(-n)_\nu,$$

where

$$(17) \quad C'_\nu = \lim_{b_u \to \infty} \lim_{b_{u+1} \to \infty} \cdots \lim_{b_v \to \infty} (C_\nu / b_u b_{u+1} \cdots b_v), \quad u = q' + 1, \ v = p + 1,$$

and $P''_n(x)$ is $P_n(x)$ with q replaced by q'. This limit process applied to (11) gives

$$(18) \qquad -(k + 1)(b_q + k) = \sum_{\nu=0}^{\sigma} C'_\nu \alpha_\nu(k + 1)\beta_\nu(k + 1),$$

and (12) is used unchanged for D_ν. These two equations, though, are just (11) and (12) for $p + 1 > q$.

Thus (11) and (12) are established for all p, q and we have succeeded in "uncoupling" Eq. (7) to give Eqs. (11) and (12), which involve C_ν and D_ν alone, respectively.

Next, we solve these two equations.

In Eq. (11), let $k + 1 - n = -s$, $s = 0, 1, 2, \cdots, \sigma$. The result can be written

$$(19) \quad \sum_{\nu=0}^{s} \frac{(-)^\nu C_\nu (-s)_\nu}{(s + 1 - 2n - \lambda)_\nu} = \epsilon(n - s) + \frac{(-)^{\sigma+1}(n - s)(n + b_q - s - 1)}{(s + 1 - 2n - \lambda)_{\sigma - s}},$$

$$s = 0, 1, 2, \cdots, \sigma.$$

But if $1 - 2n - \lambda \neq 0, -1, -2, \cdots$, the above equation can be solved for C_ν by applying a lemma of Wimp [1]. After some algebra and evaluation of ${}_2F_1$'s of unit argument, one arrives at (5). To find the D_ν's, let $k - n = -s$ in (12) and proceed in a similar fashion.

III. Concluding Remarks. If $p + 1 = q$ and $x = 1$ in (3), we get a recursion relation for $P_n(1)$ of order $\max(p + 1, q)$. Note that this is of order one less than that obtained by putting $x = 1$ in the homogeneous linear difference equation satisfied by $P_n(x)$ given in [1]. If $p = 1$, the resulting recursion relation for

$$(20) \qquad {}_3F_2\left(\begin{matrix} -n, n + \lambda, a_1 \\ b_1, b_2 \end{matrix} \middle| \, 1\right)$$

is that given by Bailey [3], which in turn is Watson's result [2] slightly rewritten. For $p + 1 = q$ and x general, (3) of course provides a generalization of the classical differential-difference formula for the Jacobi polynomials, see [4, p. 170 (15)].

A differential-difference relation for the polynomials

$$(21) \qquad Q_n(x) = {}_{p+1}F_q\left(\begin{matrix} -n, a_1, a_2, \cdots, a_p \\ b_1, b_2, \cdots, b_q \end{matrix} \middle| \, x\right)$$

can easily be obtained from (3) by replacing x by x/λ and letting $\lambda \longrightarrow \infty$.

We point out that the conditions of the theorem can be relaxed considerably. If λ is a positive integer m, we can write

$$(-n)_\nu/(1 - n - \lambda)_\nu = n!(n + 1 - \nu)_{m-1}/\Gamma(n + m), \tag{22}$$

which is well defined for all n, so condition (i) is not essential to the analysis.

Also, if any of the quantities (ii) are negative integers or zero, limits may be taken after the equation has been multiplied by a suitable factor, see [1]. The quantity n can even be nonintegral when $q > p + 1$ or when $q = p + 1$ and $|\arg(1 - x)| < \pi$, by the permanence principle for functional equations. (It may be necessary, in this case, to multiply the equation by a factor $(r - n - \lambda)$ to make the coefficients well defined.)

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On the Computation of Tricomi's Ψ Function

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Abstract — Zusammenfassung

On the Computation of Tricomi's Ψ Function. Two methods for calculating Tricomi's confluent hypergeometric function are discussed. Both methods are based on recurrence relations.

The first method converges like

$$\exp(-\alpha|\lambda|^{1/3} n^{2/3}) \text{ for some } \alpha>0$$

and the second like

$$\exp(-\beta|\lambda|^{1/2} n^{1/2}) \text{ for some } \beta>0.$$

Several examples are presented.

Über die Berechnung von Tricomis Ψ-Funktion. Zwei verschiedene Methoden, um Tricomis Confluente Hypergeometrische Funktion $\psi(a,\sigma;\lambda)$ zu berechnen werden angegeben. Die beiden Methoden gründen sich auf den Gebrauch rekurrenter Relationen.

Die erste Methode konvergiert wie

$$\exp(-\alpha|\lambda|^{1/3} n^{2/3}), \text{ für ein bestimmtes } \alpha>0$$

und die zweite wie

$$\exp(-\beta|\lambda|^{1/2} n^{1/2}), \text{ für ein bestimmtes } \beta>0.$$

In dieser Arbeit werden Beispiele hierfür vorgestellt.

1. Introduction

In this paper we propose two methods for computing Tricomi's confluent hypergeometric function $\Psi(a,\sigma;\lambda)$. Both methods use the Miller algorithm (computation by backward recursion) as applied to difference equations satisfied by sequences of related functions. The first method utilizes a third order difference scheme with convergence like $e^{-\alpha|\lambda|^{1/3}n^{2/3}}$ for some $\alpha>0$ and the second scheme uses a second order difference equation with somewhat inferior convergence of $e^{-\beta|\lambda|^{1/2}n^{1/2}}$, $\beta>0$.

Our notation for special functions in this paper and the source of their elementary properties is always that of the Erdélyi *et al.* volumes, reference [1]. For a discussion of the Miller algorithm, the reader may consult any of a number of papers, see e.g., [2], [3], [4], and [5] and the references given in these.

2. Third Order Difference Scheme

We define

$$U_n \equiv U_n(a,b,c;\lambda)=\frac{1}{2\pi i}\int_L \mu(s)\,\frac{\Gamma(n+s)}{\Gamma(n+a-s)}\,\lambda^s\,ds,$$
$$\mu(s)=\frac{\Gamma(a-s)\,\Gamma(b-s)\,\Gamma(c-s)}{\Gamma(b)\,\Gamma(c)},\quad n=0,1,2,\ldots,\ \lambda\neq 0, \tag{2.1}$$

where a, b, c are complex constants and L is a path going from $-i\infty$ to $i\infty$ such that the poles of $\mu(s)$ are to the right, those of $\Gamma(n+s)$ to the left of the contour. If none of a, b, c are negative integers or zero, this is always possible, and this assumption will be maintained throughout. Then U_n is analytic in a, b, c. Further, the integral converges for $|\arg\lambda|<\frac{3\pi}{2}$. See [1] for a treatment of such integrals. We shall use liberally the results of chapters 5 and 6 of that reference. In the usual notation of Meijer's G-function, we have

$$U_n=G_{23}^{31}\left(\lambda\,\middle|\,\begin{matrix}1-n,\ n+a\\ a, b, c\end{matrix}\right)\Big/\Gamma(b)\,\Gamma(c) \tag{2.2}$$

and further

$$U_n+(C_1+\lambda D_1)\,U_{n+1}+(C_2+\lambda D_2)\,U_{n+2}+C_3\,U_{n+3}=0 \tag{2.3}$$

$n=0, 1, 2, \ldots$, where

$$\begin{aligned}
C_1&=(2n+a+1)\left[1-\frac{(2n+a)\,\tau_{n+1}}{(2n+a+3)\,\tau_n}\right],\\
C_2&=\frac{(2n+a)}{(2n+a+4)\,\tau_n}\,[(n+2)(2n+a+1)(n+a+2-c)(n+a+2-b)\\
&\qquad -(n+a+1-b)(n+a+1-c)(n+1)(2n+a+4)],\\
C_3&=-\frac{(n+2)(2n+a)(2n+a+1)(n+a+2-b)(n+a+2-c)}{\tau_n\,(2n+a+3)(2n+a+4)},\\
D_1&=-(2n+a)(2n+a+1)/\tau_n=D_2,\\
\tau_n&=(n+a)(n+b)(n+c).
\end{aligned} \tag{2.4}$$

Equation (2.3) may be verified by substituting (2.3) in (2.1), splitting the integral up into two parts, one with λ^s, the other with λ^{s+1}, letting $s\to s-1$ in the latter, deforming the path of integration and recombining. This requires additional assumptions about the separation of the poles of the integrand of (2.1), but after (2.3) is established for a, b, c in certain regions, it holds for all a, b, c not zero or negative integers by the permanence principle of functional equations [6]. The work in [7] shows

$$U_n\sim C_0(\lambda)\,n^\mu\,e^{-3(n^2\lambda)^{1/3}}\,S(n^{1/3}),\ n\to\infty,\ |\arg\lambda|<\frac{3\pi}{2},$$

$$C_0(\lambda)=\frac{2\pi\lambda^{\frac{a+b+c-1}{3}}e^{\frac{\lambda}{3}}}{3^{1/2}\,\Gamma(b)\,\Gamma(c)},\ \mu=-\frac{a+2b+2c-2}{3} \tag{2.5}$$

and $S(z)$ is an asymptotic series of Poincaré type in powers of $1/z$.

The Birkhoff-Trjitzinsky analytic theory of difference equations [8], [9] asserts that there are two more linearly independent solutions of (2.3) considered as a difference equation in n which having the behaviour

$$\begin{gathered}U_n^{(h)}\sim n^{\mu}\,e^{-3(n^2\lambda)^{1/3}\omega_h}\,S(n,\omega_h),\ n\to\infty,\ |\arg\lambda|<\frac{3\pi}{2},\\ h=1,2,\ \omega_h=e^{\frac{4h\pi i}{3}}.\end{gathered} \tag{2.6}$$

We also have

$$\begin{gathered}\frac{\Gamma(a)}{\Gamma(c)}(\lambda x)^a\,\Psi(a,a+1-b;\lambda x)=\sum_{n=0}^{\infty}\frac{(2n+a-1)\,\Gamma(n+a-1)(-1)^n}{\Gamma(n+c)}\\ \times U_n\,R_n^{(a-c-1,c-1)}\left(\frac{1}{x}\right),\ 1\le x\le\infty,\ \lambda\neq 0,\ |\arg\lambda|<\frac{3\pi}{2}\end{gathered} \tag{2.7}$$

where $R_n^{(\alpha,\beta)}$ denotes the shifted Jacobi polynomial, see [10]. Letting $x\to\infty$ and using the known asymptotic formula for Ψ gives the normalization relation

$$1=\sum_{n=0}^{\infty}\frac{(2n+a-1)\,\Gamma(n+a-1)}{n!}\,U_n/\Gamma(a) \tag{2.8}$$

and a special provision must be made if $a=1$. The first coefficient enters with half weight and

$$\begin{gathered}1=\sum_{n=0}^{\infty}\varepsilon_n\,U_n^*/\Gamma(a),\ U_n^*=U_n|_{a=1},\\ \varepsilon_n=1,\ n=0,\\ =2,\ n>0.\end{gathered} \tag{2.9}$$

Putting $n=0$ gives us our desired function, namely

$$\begin{aligned}U_0&=G_{12}^{21}\left(\lambda\,\middle|\,\begin{matrix}1\\ b,c\end{matrix}\right)/\Gamma(b)\,\Gamma(c)\\ &=\lambda^b\,\Psi(b,\sigma;\lambda),\ \sigma=b+1-c.\end{aligned} \tag{2.10}$$

The quantity a thus appears as an arbitrary parameter in the computation of U_0 from (2.3) and the normalization relation (2.8). If $a=1$, the coefficients C_j, D_j simplify considerably and

$$\begin{gathered}U_n^*+(C_1^*+\lambda D_1^*)\,U_{n+1}^*+(C_2^*+\lambda D_2^*)\,U_{n+2}^*+C_3^*\,U_{n+3}^*=0\\ n=0,1,2,\ldots,\end{gathered} \tag{2.11}$$

where

$$
\begin{aligned}
C_1^* &= (2n+2)\left[1-\frac{(2n+1)\gamma_{n+1}}{2(n+1)\gamma_n}\right], \\
C_2^* &= \frac{(2n+1)}{(2n+5)\gamma_n}\left[(2n+4)(n+3-c)(n+3-b)-(n+2-b)(n+2-c)(2n+5)\right] \\
C_3^* &= -\frac{(2n+1)(n+3-b)(n+3-c)}{(2n+5)\gamma_n} \\
D_1^* &= -2(2n+1)/\gamma_n = D_2^*,\ \gamma_n=(n+b)(n+c).
\end{aligned}
\tag{2.12}
$$

Theorem 1:
Let none of a, b, c be negative integers or zero, $\lambda \neq 0$, $a \neq 1$, $|\arg \lambda| < \pi$. Let M be an integer ≥ 0. Define

$$U_{M+2}^M = U_{M+1}^M = 0,\ U_M^M = 1 \tag{2.13}$$

and compute U_n^M for $0 \leq n \leq M-1$ from (2.3) by substituting U_n^M for U_n. Let

$$W_M = \sum_{n=0}^{M} \frac{(2n+a-1)\,\Gamma(n+a-1)}{n!}\, U_n^M/\Gamma(a). \tag{2.14}$$

Then

$$\lim_{M\to\infty} \frac{U_n^M}{W_M} = U_n,\ n=0,1,2,\ldots. \tag{2.15}$$

Also, U_n^* may be computed in the same manner from (2.9)–(2.11) with the same restrictions on b, c, λ.

Proof: It is convenient to use the notation

$$\psi_n = \mathscr{P}(\phi_n),\ \ n\to\infty \tag{2.16}$$

if

$$\psi_n = \mathcal{O}(n^\alpha \phi_n)\ \text{ for some } \alpha,\ n\to\infty \tag{2.17}$$

We require the fact [11] that if

$$S(n) = \sum_{k=0}^{n} f(k),\ \ n=0,1,2,\ldots. \tag{2.18}$$

$$f(n) = e^{\alpha n^\beta}\, n^\theta\,[1+0(1)],\ \ n\to\infty. \tag{2.19}$$

Then

$$\left.\begin{aligned} S(n) &= A + \mathscr{P}(e^{\alpha n^\beta}),\ Re\,\alpha<0, \\ A &= S(\infty) \end{aligned}\right\} \tag{2.20}$$

and

$$S(n) = (\mathscr{P}(e^{\alpha n^\beta}),\ \ Re\,\alpha>0. \tag{2.21}$$

Let

$$y_1(n) = U_n \tag{2.22}$$

and let y_2, y_3 be two other linearly independent solutions of (2.3) corresponding to $h = 1, 2$ in (2.6) respectively. Let

$$S_h(n) = \sum_{k=0}^{n} \frac{(2k+a-1)\,\Gamma(k+a-1)}{k!}\, y_h(k)/\Gamma(a). \tag{2.23}$$

Then by expressing U_n^M as a linear combination of y_1, y_2, y_3 and substituting in (2.13) we find

$$\left.\begin{aligned}
&\frac{U_n^M}{W_M} = \frac{T_1(M)\,y_1(n) + T_2(M)\,y_2(n) + T_3(M)\,y_3(n)}{T_1(M)\,S_1(M) + T_2(M)\,S_2(M) + T_3(M)\,S_3(M)},\\
&T_1(M) = \begin{vmatrix} y_2(M+1) & y_3(M+1) \\ y_2(M+2) & y_3(M+2) \end{vmatrix}, \quad T_2(M) = -\begin{vmatrix} y_1(M+1) & y_3(M+1) \\ y_1(M+2) & y_3(M+2) \end{vmatrix},\\
&T_3(M) = \begin{vmatrix} y_1(M+1) & y_2(M+1) \\ y_1(M+2) & y_2(M+2) \end{vmatrix}.
\end{aligned}\right\} \tag{2.24}$$

We can write

$$\frac{U_n^M}{W_M} = \frac{y_1(n) + V_1(M)\,y_2(n) + V_2(M)\,y_3(n)}{1 + V_1^*(M) + V_2^*(M) + V_3^*(M)} \tag{2.25}$$

where

$$\left.\begin{aligned}
&V_h(M) = [e^{-3(M^2\lambda)^{1/3}(1-\omega_h)}],\\
&V_h^*(m) = [e^{3(M^2\lambda)^{1/3}\omega_h}],\ h = 1, 2\\
&V_3^*(M) = [e^{-3(M^2\lambda)^{1/3}(1-\omega_1-\omega_2)}];
\end{aligned}\right\} \tag{2.26}$$

and

$$\begin{gathered}
\frac{U_n^M}{W_M} - U_n = \mathcal{P}\,(e^{3(M^2|\lambda|)^{1/3}\kappa}), \quad M \to \infty\\
\kappa = \max_{h=1,2} \cos\left[\frac{\arg\lambda}{3} + \frac{4\pi h}{3}\right] < 0, \quad |\arg\lambda| < \pi,
\end{gathered} \tag{2.27}$$

and this not only establishes the theorem but gives a convergence estimate.

3. The Case $a \to \infty$, Second Order Scheme

If we let $a \to \infty$ in $a^n\, U_n/n!$ we have, by dominated convergence and

$$\Gamma(a+\alpha)/\Gamma(a+\beta) = a^{\alpha-\beta}\,[1 + 0(a^{-1})], \quad |\arg a| < \pi, \tag{3.1}$$

the result

$$\left.\begin{aligned}
V_n &\equiv V_n(b,c;\lambda) = \lim_{a\to\infty} \frac{a^n U_n}{n!} = \frac{1}{2\pi i n!} \int_L \Gamma(b-s)\,\Gamma(c-s) \\
&\quad \times \Gamma(n+s)\lambda^s\, ds/n!\,\Gamma(b)\,\Gamma(c) \\
&= G_{12}^{21}\left(\lambda \,\middle|\, \begin{matrix} 1-n \\ b,c \end{matrix}\right)/n!\,\Gamma(b)\,\Gamma(c) \\
&= \frac{\lambda^b (b)_n (c)_n}{n!}\,\Psi(n+b,\sigma;\lambda),\ \sigma = b+1-c,
\end{aligned}\right\} \tag{3.2}$$

and the recursion relation becomes very simple indeed:

$$V_n - \frac{(n+1)\,[(2n+b+c+1)+\lambda]}{(n+b)\,(n+c)}\,V_{n+1} + \frac{(n+1)\,(n+2)}{(n+b)\,(n+c)}\,V_{n+2} = 0. \tag{3.3}$$

The normalization relationship

$$1 = \sum_{n=0}^{\infty} V_n \quad |\arg\lambda| < \pi \tag{3.4}$$

is best determined by using the standard integral representation for V_n, see [1 (I), p. 256 (3)] and summing under the integral sign, which is permissible for the range of values of λ considered.

If we seek asymptotic representations for the solutions of (3.3) following the methods of [8], [9] we find the equation has two linearly independent solutions which have the asymptotic behaviour

$$y_k^{(h)} = d_h(\lambda)\, n^{\frac{b+c}{2}-\frac{5}{4}}\, e^{\mp 2(n\lambda)^{1/2}}\, S(\pm n^{1/2}),\ n\to\infty \tag{3.5}$$

where for $h=1$ we take the upper, otherwise the lower, sign and $S(z)$ is a Poincaré asymptotic series. The estimates given in Slater [12] allow us to identify $y_1(n)$ with V_n for $|\arg\lambda| < \pi$, and show

$$d_1(\lambda) = \pi^{1/2}\,\lambda^{\frac{b+c}{2}-1/4}\, e^{\frac{\lambda}{2}}/\Gamma(b)\,\Gamma(c). \tag{3.6}$$

Both for this and for the previous case, the work of Wimp, Luke and Fields provides a bases of solutions of the difference equation, see [13], [14]. We may identify $y_2(n)$ with a constant multiple of

$$\frac{(b)_n}{n!}\,\Phi(n+b,\sigma;\lambda). \tag{3.7}$$

To apply the Miller algorithm we define as sequence V_n^M by

$$V_{M+1}^M = 0,\ V_M^M = 1, \tag{3.8}$$

and V_n^M is then computed from (3.3) for $0 \le n \le M-1$. Then

$$\left.\begin{aligned}\frac{V_n^M}{W_M} &= V_n + \mathscr{P}\left(e^{-2(M\lambda)^{1/2}}\right),\ M \to \infty, \\ W_M &= \sum_{k=0}^{M} V_k^M,\end{aligned}\right\} \tag{3.9}$$

as an easy computation shows. We now have

Theorem 2:
Let neither b nor c be a negative integer or zero, $\sigma = b+1-c$, $\lambda \neq 0$, $|\arg \lambda| < \pi$. Then

$$\lim_{M\to\infty} \frac{V_n^M}{W_M} = V_n = \frac{\lambda^b (b)_n (c)_n}{n!}\, \Psi(n+b, \sigma; \lambda). \tag{3.10}$$

To illustrate, take $b=c=1/2$,

$$\begin{aligned} V_n &= \frac{\lambda^{1/2} \left(\frac{1}{2}\right)_n^2}{n!}\, \Psi(N+1/2, 1; \lambda) \\ &= \frac{\lambda}{n!} \frac{d^n}{d\lambda^n} \left[\lambda^{n-1/2}\, \Psi(1/2, 1; \lambda)\right] \qquad (3.11) \\ &= \frac{\lambda}{\sqrt{\pi}\, n!} \frac{d^n}{d\lambda^n} \left[\lambda^{n-1/2} e^{\frac{\lambda}{2}} K_0\left(\tfrac{\lambda}{2}\right)\right]. \end{aligned}$$

By using a generating function, we may express V_n in terms of $K_0\left(\frac{\lambda}{2}\right), K_1\left(\frac{\lambda}{2}\right), \ldots, K_n\left(\frac{\lambda}{2}\right)$. Since

$$K_\nu\left(\tfrac{\lambda}{2}\right) = \frac{1}{2} \int_0^\infty e^{-\frac{\lambda}{4}\phi}\, t^{-\nu-1}\, dt,\ \phi = t + t^{-1},\ Re\,\lambda > 0, \tag{3.12}$$

we see

$$V_n(\lambda) = \frac{\lambda}{\sqrt{\pi}\, n!\, 2} \int_0^\infty t^{-1} \frac{d^n}{d\lambda^n} \left[e^{\frac{\lambda}{2} - \frac{\lambda}{4}\phi}\, \lambda^{n-1/2}\right] dt \tag{3.13}$$

and by using the Rodrigues representation of the following Laguerre polynomial

$$L_n^{(-1/2)}(x) = \frac{\left(\frac{1}{2}\right)_n}{n!}\, \Phi\left(-n, \tfrac{1}{2}; x\right) \tag{3.14}$$

we find that

$$\left.\begin{aligned} V_n(\lambda) &= \frac{\lambda^{\frac{1}{2}} e^{\frac{\lambda}{2}}}{\sqrt{\pi}} \sum_{r=0}^{n} \varepsilon_r\, c_{n,r}(\lambda)\, K_r\left(\tfrac{\lambda}{2}\right), \\ \varepsilon_r &= 1,\ r = 0 \\ &= 2,\ r > 0. \end{aligned}\right\} \tag{3.15}$$

The $c_{n,r}(\lambda)$ are determined by

$$L_n^{(-1/2)}\left[-\tfrac{\lambda}{2}+\tfrac{\lambda}{4}(t+t^{-1})\right]=c_{n,0}(\lambda)+\sum_{r=1}^{n} c_{n,r}(\lambda)(t^r+t^{-r}). \tag{3.16}$$

By analytic continuation (3.15) holds for all $\lambda \neq 0$ when the Riemann surfaces for the functions involved are defined appropriately.

In particular

$$V_0=\frac{\lambda^{\frac{1}{2}} e^{\frac{\lambda}{2}}}{\sqrt{\pi}} K_0(\lambda/2),\ V_1=\frac{\lambda^{\frac{3}{2}} e^{\frac{\lambda}{2}}}{2\sqrt{\pi}}\left[(1+\tfrac{1}{\lambda}) K_0(\tfrac{\lambda}{2})-K_1(\tfrac{\lambda}{2})\right]. \tag{3.17}$$

For $\lambda=4$ standard tables give

$$V_0=.949608042\ldots,\quad V_1=.041712616\ldots\ldots \tag{3.18}$$

Taking $M=10$ gives

$$V_0^{10}=.949611302\ldots,\quad V_1^{10}=.041712759\ldots, \tag{3.19}$$

with approximate errors of 3.3×10^{-6} and 1.4×10^{-7} respectively.

It is interesting that the difference equation (3.3) serves to compute K_0 while the usual recursion relation

$$K_n(z)+\frac{2(n+1)}{z} K_{n+1}(z)-K_{n+2}(z)=0,\ n=0,1,2,\ldots, \tag{3.20}$$

does not provide a stable algorithm when used in the backward direction to compute $K_0, K_1, K_2, \ldots$. Once K_0 is computed from (3.3), K_1 can be found from (3.17) and then (3.20) can be used stably in the *forward direction* to compute K_n for all n.

For the general case, despite the simplicity of the algorithm of Theorem 2, equation (3.9) compared with equation (2.27) shows the convergence to be inferior to the more complicated scheme of Theorem 1. In some situations, however, all the functions $\Psi(n+b,\sigma;\lambda)$, $n=0, 1, 2, \ldots$ are desired, so the latter scheme is the more desirable.

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Toeplitz Arrays, Linear Sequence Transformations and Orthogonal Polynomials

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Summary. Let s_n be a given sequence, $U=[\mu_{i,j}]_{i,j=0}^{\infty}$ an infinite array of complex numbers $\mu_{i,j}=0,\ i<j,\ \sum_{k=0}^{i}\mu_{i,k}=1$, and $s_{n,m}=\sum_{k=0}^{m}\mu_{m,k}s_{n+k}$. We develop and discuss linear sequence transformations defined by

$$s_{n,m+1}=a_m s_{n+1,m}+b_m s_{n,m}$$

or

$$s_{n,m+1}=a_m s_{n,m}+b_m s_{n+1,m}+c_m s_{n,m-1},$$

$n, m=0, 1, 2, \ldots$. We ask, if $s_n\to\alpha$ as $n\to\infty$, does $s_{n,m}\to\alpha$ as $m\to\infty$? This convergence and the rapidity of it are seen to depend on the location of the zeros of the polynomial $P_m(\lambda)=\sum_{k=0}^{m}\mu_{m,k}\lambda^k$. When the P_m are chosen to be certain polynomials encountered in special function theory, such as Bessel polynomials or Legendre polynomials, the result is an elegant and powerful set of algorithms for improving the convergence of large classes of sequences.

I. Background

The infinite array

$$U=[\mu_{n,k}]=\begin{bmatrix}\mu_{00} & & (0)\\ \mu_{10} & \mu_{11} & \\ \mu_{20} & \mu_{21} & \mu_{22}\\ \vdots & \vdots & \vdots\end{bmatrix} \tag{1.1}$$

$$\mu_{n,k}\ \text{complex},\quad \sum_{k=0}^{n}\mu_{n,k}=1,$$

is called a Toeplitz array. Given a sequence $\{s_n\}$ of complex numbers we define a transformation

$$U:\{s_n\}\to\{\bar{s}_n\}, \tag{1.2}$$

$$\bar{s}_n=\sum_{k=0}^{n}\mu_{n,k}s_k,\quad n=0,1,2,\ldots. \tag{1.3}$$

Let X be the space of all sequences, X_C the subspace of convergent sequences, S a subspace of X.

U is *totally regular* if

$$\lim_{n\to\infty}s_n=\lim_{n\to\infty}\bar{s}_n\quad \forall\,\{s_n\}\in X_C. \tag{1.4}$$

U is *regular for* S if (1.4) holds for all $\{s_n\} \in S \subset X_C$. U is *extensive for* S if $\lim_{n\to\infty} \bar{s}_n$ exists for all $\{s_n\} \in S \subset X - X_C$. U is *effective for* S if $\lim_{n\to\infty} \bar{s}_n$ exists for all $\{s_n\} \in S$ and (1.4) holds whenever $\{s_n\} \in X_C$. Finally, U is *accelerative for* S if

$$\lim_{n\to\infty} \frac{|\bar{s}_n - \alpha|}{|s_n - \alpha|} = 0 \tag{1.5}$$

for $\{s_n\} \in S \subset X_C$, $\lim_{n\to\infty} s_n = \alpha$.

The polynomial

$$P_n(\lambda) = \sum_{k=0}^{n} \mu_{n,k} \lambda^k \tag{1.6}$$

we call the nth *characteristic polynomial* of U.

There is an enormous literature on the properties of Toeplitz arrays. The books of Hardy [1] and Knopp [2] contain some material, and the book of Cooke [3] is almost entirely devoted to the subject. It provides an excellent bibliography of the classical aspects of the problem at least up to 1950. Modern approaches are discussed briefly in Taylor [4]. For references between 1950 and the present the best source is Mathematical Reviews. Most writers have concentrated on describing properties of the $\mu_{n,k}$ which will relate properties of $\{s_n\}$ to $\{\bar{s}_n\}$, for example, how limit points in $\{s_n\}$ are affected by the transformation, whether U preserves convergence or boundedness, etc.

Our discussion in this paper will be rather more practically oriented, and will center around the polynomial $P_n(\lambda)$. We will show how certain choices of $P_n(\lambda)$ from among the polynomials of special function theory produce transformations U which lead to efficient and simple algorithms for improving the convergence of important classes of sequences.

To begin our investigation, we need some results about the regularity of U.

Lemma 1. Let $\{\lambda_k\}_{k=1}^{\infty}$ be an infinite sequence of real positive numbers none of which equals one. Let

$$P_n(\lambda) = Q_n(\lambda)/Q_n(1), \tag{1.7}$$

$$Q_n(\lambda) = \prod_{k=1}^{n} (\lambda - \lambda_k). \tag{1.8}$$

If the series

$$\sum \frac{\lambda_k}{1 - \lambda_k} \tag{1.9}$$

converges, then $[\mu_{n,k}]$ is totally regular.

Proof. Since $\mu_{n,k}$ is the residue of $z^{-k-1} P_n(z)$ at the origin, we have

$$\begin{aligned}\mu_{n,k} &= \frac{1}{2\pi i} \int_{(0^+)} z^{-k-1} \prod_{j=1}^{n} \left(\frac{z - \lambda_j}{1 - \lambda_j}\right) dz \\ &= \frac{1}{2\pi i} \int_{|t|=R} t^{k-n-1} \prod_{j=1}^{n} \left(\frac{1 - t\lambda_j}{1 - \lambda_j}\right) dt.\end{aligned} \tag{1.10}$$

So

$$|\mu_{n,k}| \leqq R^{k-n} \sup_{|t|=R} \left| \prod_{j=1}^{n} \left(\frac{1-t\lambda_j}{1-\lambda_j} \right) \right|. \tag{1.11}$$

By Knopp [2, p. 224] the product

$$M(t) = \prod_{j=1}^{\infty} \left(\frac{1-t\lambda_j}{1-\lambda_j} \right) \tag{1.12}$$

converges for all $t \neq 1$ provided $\sum \frac{\lambda_k}{1-\lambda_k}$ and $\sum \left(\frac{\lambda_k}{1-\lambda_k} \right)^2$ converge.

But the convergence of the first series implies $\lambda_k \to 0$. Since $\lambda > 0$, the second series converges also. Letting $R > 1$, $n \to \infty$ in (1.11) shows $\mu_{n,k} \to 0$. Also, since all λ_k are positive, we have

$$(-1)^n P_n(-1) = \sum_{k=0}^{n} (-1)^{n-k} \mu_{n,k} = \sum_{k=0}^{n} |\mu_{n,k}| = \left| \prod_{k=1}^{n} \left(\frac{-1-\lambda_k}{1-\lambda_k} \right) \right| \leqq M(-1) \tag{1.13}$$

since the product on the right is nondecreasing in n. Thus $[\mu_{n,k}]$ is totally regular (See Knopp [2, p. 391]).

Example. Let $\lambda_k = \sigma^{-k}$, $\sigma > 1$. Then $[\mu_{n,k}]$ are the weights for the Romberg integration procedure, see [5].

A simpler statement can be made when all the zeros of $P_n(\lambda)$ are negative.

Lemma 2. Let $P_n(\lambda)$ be as in (1.7) but with $\lambda_k = \lambda_{n,k} \in [-a, 0]$, $a > 0$. Then $[\mu_{n,k}]$ is totally regular.

Proof. As before

$$|\mu_{n,k}| \leqq R^k \prod_{j=1}^{n} \left(\frac{\frac{1}{R} - \lambda_j}{1-\lambda_j} \right). \tag{1.14}$$

But $\left(\frac{1}{R} + x \right) \Big/ (1+x)$ is monotone increasing in x, so

$$|\mu_{n,k}| \leqq R^k \left(\frac{\frac{1}{R} + a}{1+a} \right)^n \to 0 \quad \text{as} \quad n \to \infty \tag{1.15}$$

if $R > 1$. Also all the $\mu_{n,k}$ are positive so

$$\sum_{k=0}^{n} |\mu_{n,k}| = P_n(1) = 1 \tag{1.16}$$

and the lemma is shown.

A particular case is when $\{P_n(\lambda)\}$ are a set orthogonal on $[-a, 0]$ with respect to a weight function $\omega(\lambda)$ that is positive, $\not\equiv 0$ and integrable, then $[\mu_{n,k}]$ is totally regular. If $\omega(\lambda) = \lambda^{-1/2}(1+\lambda)^{-1/2}$ then

$$\mu_{n,k} = \frac{(n)_k}{(\frac{1}{2})_k} \binom{n}{k} \Big/ Q_n(1), \tag{1.17}$$

$$(b)_k = b(b+1) \ldots (b+k-1)$$

and $Q_n(\lambda)$ is the Chebyshev polynomial of degree n normalized to $[-1, 0]$. The weights above are among the most useful of all Toeplitz methods, as we shall see.

There are also some non-regular cases of practical importance.

Let

$$\mu_{n,k} = \frac{(\tau+k)_n(-1)^{n+k}}{k!\,(n-k)!}, \qquad \tau > 0. \tag{1.18}$$

Then

$$P_n(\lambda) = (-1)^n P_n^{(\tau-1,0)}(1-2\lambda) \tag{1.19}$$

and the zeros of $P_n(\lambda)$ lie in $[0, 1]$. Since the k-th zero depends on n, U is not covered by Lemma 1. In fact, U is not totally regular. Let

$$s_n = (-\sigma)^n, \qquad 0 < \sigma < 1. \tag{1.20}$$

Then

$$|\bar{s}_n| = \frac{1}{n!}\sum_{k=0}^{n} \sigma^k (\tau+k)_n \binom{n}{k} > \frac{\sigma^n(\tau+n)_n}{n!} > C\,2^{2n} n^{-1/2}\sigma^n, \qquad n > n_0. \tag{1.21}$$

Taking any value of $\sigma > \frac{1}{4}$ shows U is not totally regular. But U is still useful for summing certain types of sequences. Let S be the space of all sequences of the form

$$s_n = \alpha + \frac{c_1}{(n+\tau)} + \frac{c_2}{(n+\tau)_2} + \frac{c_3}{(n+\tau)_3} + \cdots, \qquad n = 0, 1, 2, \ldots, \qquad \tau > 0. \tag{1.22}$$

Theorem 1. Let

$$|c_n| \leqq \varrho^n m_n, \qquad n = 1, 2, \ldots, \tag{1.23}$$

where m_n is a non-increasing sequence.

Then

$$|\bar{s}_n - \alpha| \leqq \frac{m_{n+1}(n-1)!}{(2n)!}\varrho^{n+1} e^{\varrho}, \qquad n = 1, 2, \ldots, . \tag{1.24}$$

Proof. From here on we will use the notation

$$r_n = s_n - \alpha, \qquad \bar{r}_n = \bar{s}_n - \alpha \tag{1.25}$$

for the remainders of the sequences s_n, $\bar{s}_n$ when they converge. We have

$$\bar{r}_n = \frac{(-1)^n \Gamma(n+\tau)}{n!}\sum_{t=1}^{\infty} \frac{c_t \Gamma(t)}{\Gamma(t+\tau+n)\Gamma(t-n)} \tag{1.26}$$

which results by using the known formula for ${}_2F_1(1)$. Thus

$$|\bar{r}_n| \leqq \frac{1}{n!}\sum_{t=0}^{\infty} \frac{|c_{t+n+1}|(n+t)!}{(n+\tau)_{n+t+1}t!}. \tag{1.27}$$

Now $(n+\tau)_{n+t+1}$ is increasing in τ. We majorize the sum on the right by setting $\tau = 0$ and using (1.23).

$$|\bar{r}_n| \leqq \frac{m_{n+1}\varrho^{n+1}\Gamma(n)}{\Gamma(2n+1)}\,\Phi(n+1, 2n+1|\varrho). \tag{1.28}$$

Φ being Tricomi's Φ-function. Each of the terms in the series of this function are decreasing in n. Letting $n = 0$ gives an upper bound, and the Theorem results.

Table 1. $\mu_{n,k}$

n	k 0	1	2	3	4	5	6	7
0	1							
1	−1	2						
2	1	−6	6					
3	−1	12	−30	20				
4	1	−20	90	−140	70			
5	−1	30	−210	560	−630	252		
6	1	−42	420	−1680	3150	−2772	924	
7	−1	56	−756	4200	−11550	16632	−12012	3432

It is clear that if $c_j=0$, $j\geqq j_0$, then $\bar{s}_n$ is exact for $n>j_0$. As can be seen by a simple rearrangement of series, U is useful for improving convergence of sequences of the form

$$s_n=\alpha+\sum_{j=1} d_j n^{-j}. \tag{1.29}$$

The value of τ used is not really important. When $\tau=1$, the $P_m(\lambda)$ are Legendre polynomials on $[0, 1]$. $\bar{s}_n$ is best computed by the techniques of section III. We give a short table of $\mu_{n,k}$ for $\tau=1$.
As an example, we compute Euler's constant from the sequence

$$s_n=\sum_{r=1}^{n}\frac{1}{r}-ln(n+1)=\gamma+\sum_{s=1}^{\infty}\frac{c_s}{(n+1)_s} \tag{1.30}$$

$$\gamma=0.577215\ldots$$

see Nielsen [6, p. 84 ff.]. We have the following table.

Table 2

n	s_n	$\bar{s}_n$	r_n
0	0	0	-5.77×10^{-2}
1	0.306852820	0.613705640	3.65×10^{-2}
2	0.401387711	0.567209346	1.00×10^{-2}
3	0.447038972	0.58138195	4.17×10^{-3}
4	0.473895421	0.57506099	-2.15×10^{-3}
5	0.491573864	0.5784881	1.27×10^{-3}
6	0.504089851	0.5763945	-8.21×10^{-4}

The numerical hazards of using U are clear, since the $\mu_{n,k}$ get quite large, alternate in sign, and have sum 1. We used s_n accurate to 10 places and $\bar{s}_n$ is accurate to the places given. 4 places were lost in the computation of $\bar{s}_{10}$. This seems to be an inescapable consequence of non-regular methods. Similar methods based on the form (1.29) are discussed by Wimp, see [7] and the references cited there. These methods seem to be more effective on sequences like (1.29) than the present methods (for instance, the weights given by Wimp in [7, 8] give a $\bar{r}_4$ of -1.22×10^{-4} for the previous example) but the $\mu_{n,k}$ grow even more rapidly with n than those given here.

II. Arrays and Triangular Rules

Suppose we define a double array $s_{n,m}$ by $T_m(s_n)=s_{n,m}$ where

$$\begin{aligned} s_{n,m+1}&=a_m s_{n+1,m}+b_m s_{n,m}, \qquad n,m=0,1,2,\ldots \\ s_{n,0}&=s_n. \end{aligned} \tag{2.1}$$

We wish to determine under what conditions on a_m, b_m we have vertical total regularity in $s_{n,m}$, i.e.,

$$\lim_{n\to\infty} s_{n,m}=\alpha \quad \text{if} \quad s_n\to\alpha \tag{2.2}$$

or horizontal total regularity in $s_{n,m}$, i.e.,

$$\lim_{m\to\infty} s_{n,m}=\alpha \quad \text{if} \quad s_n\to\alpha. \tag{2.3}$$

By induction we see (2.2) obtains if

$$b_m=1-a_m. \tag{2.4}$$

From here on we assume this holds.

Also by induction one sees that $s_{n,m}$ can be written out as

$$s_{n,m}=\sum_{k=0}^{m}\mu_{m,k}s_{n+k} \tag{2.5}$$

for some constants $\mu_{m,k}$. Let

$$P_m(\lambda)=\sum_{k=0}^{m}\mu_{m,k}\lambda^k. \tag{2.6}$$

Substituting (2.5) in (2.1) shows

$$\sum_{k=0}^{m+1}\mu_{m+1,k}s_{n+k}=a_m\sum_{k=0}^{m}\mu_{m,k}s_{n+k+1}+b_m\sum_{k=0}^{m}\mu_{m,k}s_{n+k}. \tag{2.7}$$

This will hold for all possible sequences if and only if

$$\mu_{m+1,k}=a_m\mu_{m,k-1}+b_m\mu_{m,k}; \qquad (\mu_{m,k}=0,\ k<0 \quad \text{or} \quad k>m). \tag{2.8}$$

Multiplying by λ^k and summing from $k=0$ to $m+1$ gives

$$P_{m+1}(\lambda)=(a_m\lambda+b_m)P_m(\lambda) \tag{2.9}$$

or, in view of (2.4) we have

$$P_m(\lambda)=\prod_{j=0}^{m-1}(a_j\lambda+b_j)=\prod_{j=0}^{m-1}a_j\prod_{j=1}^{m}(\lambda-\lambda_j), \tag{2.10}$$

$$\lambda_j=\frac{a_{j-1}-1}{a_{j-1}}. \tag{2.11}$$

Also, $P_m(1)=1$. An application of Lemma 1 yields

Theorem 2. Let $a_m \notin [0, 1]$ and let the series $\sum (a_k - 1)$ converge. Then $T_m(s_n)$ is horizontally totally regular.

It often happens, of course, that $s_{n,m}$ goes to α as $m \to \infty$ much more rapidly than it goes to α as $n \to \infty$. Then the scheme defined by (2.1) is computationally desirable. This is the case for Romberg integration, again, see [5].

III. Arrays and Rhombus Rules

Now we show how rhombus-rules can be used to calculate recursively

$$s_{n,m} = \sum_{k=0}^{m} \mu_{m,k} s_{n+k}, \qquad s_{n,0} = s_n \tag{3.1}$$

where the $\mu_{m,k}$ are the coefficients of λ^k in polynomials normalized by (1.7) which are orthogonal over $[-a, 0]$, $a > 0$.

Theorem 3. Let $\{p_m(x)\}$ be a set of polynomials orthogonal over $[-1, 1]$ with respect to the weight function $\omega(x)$ which is integrable, not-identically zero, and positive with

$$p_{m+1}(x) = (A_m x + B_m) p_m(x) - C_m p_{m-1}(x), \qquad m = 0, 1, 2, \ldots, p_{-1}(x) = 0. \tag{3.2}$$

Then the rhombus rule defined by

$$\begin{gathered} s_{n,m+1} = a_m s_{n,m} + b_m s_{n+1,m} + c_m s_{n,m-1}, \\ n, m = 0, 1, 2, \ldots, s_{n,-1} = 0,\ s_{n,0} = s_n \end{gathered} \tag{3.3}$$

where

$$\begin{aligned} a_m &= (B_m + A_m)\sigma_m/\sigma_{m+1} \\ b_m &= \frac{2}{a} A_m \sigma_m/\sigma_{m+1} \\ c_m &= -C_m \sigma_{m-1}/\sigma_{m+1}, \end{aligned} \tag{3.5}$$

$$\sigma_m = \sigma_m(a) = p_m\left(\frac{2}{a} + 1\right), \quad a > 0, \tag{3.4}$$

is totally horizontally and vertically regular,

$$\lim_{m\to\infty} s_{n,m} = \lim_{n\to\infty} s_{n,m} = \lim_{n\to\infty} s_n = \alpha. \tag{3.6}$$

Proof. Note $\sigma_m(a)$ satisfies

$$\sigma_{m+1} = \left\{A_m\left(\frac{2}{a} + 1\right) + B_m\right\} \sigma_m - C_m \sigma_{m-1}. \tag{3.7}$$

Using this shows $a_m + b_m + c_m = 1$, and thus we have vertical regularity.

To show horizontal regularity, define

$$P_m(x) = \frac{p_m\left(\frac{2}{a} x + 1\right)}{p_m\left(\frac{2}{a} + 1\right)} = \sum_{k=0}^{m} \mu_{m,k} x^k. \tag{3.8}$$

Then

$$P_{m+1}(x) = (b_m x + a_m) P_m(x) + c_m P_{m-1}(x). \tag{3.9}$$

If the right-hand side of (3.8) is substituted into this recursion relation and like powers of x are equated, we find

$$\mu_{m+1,k}=a_m\mu_{m,k}+b_m\mu_{m,k-1}+c_m\mu_{m-1,k}. \tag{3.10}$$

Induction on m shows $s_{n,m}$ in (3.3) can be written

$$s_{n,m}=\sum_{k=0}^{m}\mu_{m,k}s_{n+k}, \quad n, m=0, 1, 2, \ldots \tag{3.11}$$

and horizontal regularity follows from Lemma 2, since the zeros of $P_m(x)$ are located in $(-a, 0)$.

In practice it is more useful to compute

$$t_{n,m}=\sigma_m s_{n,m} \tag{3.12}$$

and then divide $t_{n,m}$ by σ_m to get $s_{n,m}$.

As an example consider the polynomials

$$T_m(x)=\cos(m \arccos x), \quad m=0, 1, 2, \ldots. \tag{3.13}$$

Then

$$T_{m+1}(x)=\varepsilon_m x T_m(x)-T_{m-1}(x), \quad m=0, 1, 2, \ldots, \tag{3.14}$$

$$T_{-1}=0, \ \varepsilon_m=\begin{cases}1, & m=0,\\ 2, & m>0.\end{cases} \tag{3.15}$$

Let $a=1$. Tables of Chebyshev polynomials for $[0, 1]$ show $\{\sigma_m(a)\}=\{1, 3, 17, 99, 577, 3363, 19601, 114243, \ldots\}$. Then $C_m=1$, $B_m=0$, $A_m=\varepsilon_m$ and

$$t_{n,m+1}=\varepsilon_m t_{n,m}+2\varepsilon_m t_{n+1,m}-t_{n,m-1}. \tag{3.16}$$
$$t_{n,-1}=0, \quad t_{n,0}=s_n.$$

The choice $U_m(x)=p_m(x)$ yields

$$p_{m+1}(x)=2xp_m(x)-p_{m-1}(x), \quad m=0, 1, \ldots, p_0(x)=1, \tag{3.17}$$

and so

$$t_{n,m+1}=2t_{n,m}+4t_{n+1,m}-t_{n,m-1} \tag{3.18}$$

and

$$\{\sigma_m\}=\{1, 6, 35, 204, 1189, 6930, 40391, 235416, \ldots\}.$$

For both the cases $T_m(x)$, $U_m(x)$ above we have

$$\sigma_{m+1}=6\sigma_m-\sigma_{m-1}, \quad m=1, 2, 3, \ldots. \tag{3.19}$$

In the table below we have taken $p_m=T_m$ and

$$s_n=\sum_{k=0}^{n}\frac{(-1)^k}{k+1}. \tag{3.20}$$

We display only the error table. The horizontal convergence of the algorithm is remarkable in this case. Of course, (3.18) is very adaptable to machine computations. Any choice of orthogonal polynomials other than T_m or U_m will probably

Table 3. $\varepsilon_{n,m} = s_{n,m} - \ln 2$

n \ m	0	1	2	3	4	5	6
0	3.1(−1)	−2.6(−2)	−6.9(−3)	4.6(−4)	−2.2(−5)	3.8(−6)	−3.3(−7)
1	−1.9(−1)	2.7(−2)	2.9(−3)	−5.5(−4)	3.6(−5)	−3.3(−6)	
2	1.4(−1)	−2.6(−2)	−9.9(−4)	4.6(−4)	−4.6(−5)		
3	−1.1(−1)	2.4(−2)	−9.9(−6)	−3.4(−4)			
4	9.0(−2)	−2.1(−2)	5.5(−4)				
5	−7.6(−2)	1.9(−2)					
6	6.6(−2)						

lead to more complicated coefficients in the recursion formula. Also, as we shall see, one cannot expect to do much better in accelerating the convergence of s_n by choosing any other system of orthogonal polynomials, at least for large classes of sequences.

A two dimensional scheme exists also for the nonregular methods discussed in section 1. It is particularly simple for these methods since $\sigma_m = 1$. For a given τ we have $s_{n,0} = s_n$,

$$s_{n,m+1} = a_m s_{n,m} + b_m s_{n+1,m} + c_m s_{n,m-1}, \tag{3.18}$$

$$\begin{aligned} a_m &= -(2m+\tau)(2m^2+2m\tau+\tau^2-\tau)/(m+1)(m+\tau)(2m+\tau-1) \\ b_m &= (2m+\tau)(2m+\tau+1)/(m+1)(m+\tau) \\ c_m &= -(m+\tau-1)(2m+\tau+1)/(m+1)(m+\tau)(2m+\tau-1) \end{aligned} \tag{3.19}$$

and for the Legendre polynomial case, $\tau = 1$, we have the simple algorithm

$$s_{n,m+1} = \frac{(2m+1)\,[2s_{n+1,m} - s_{n,m}] - m s_{n,m-1}}{(m+1)}. \tag{3.20}$$

$s_{n,m}$ is not horizontally totally regular, but is horizontally regular for the sequences (1.22).

IV. Regularity in Special Sequence Spaces, Orthogonal Polynomials

Let S be the space of all sequences of the form

$$s_n = \alpha + c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_M\lambda_M^n, \qquad c_j \neq 0, \tag{4.1}$$

where the c_j are complex, $\lambda_j \in \Lambda_j$, a compact subset of the complex plane not containing the origin, and M is fixed. It is important to discern the effect of U on sequences in S because many sequences that arise in practice are either members of S or behave very similarly to members of S.

We have immediately

$$\overline{s_n} = \alpha + \sum_{r=1}^{M} c_r P_n(\lambda_r). \tag{4.2}$$

Let

$$\sigma(\lambda) = \overline{\lim_{n\to\infty}} \, |P_n(\lambda)|^{\frac{1}{n}}. \tag{4.3}$$

Then given $\varepsilon > 0$ there is an n_0 such that

$$|\bar{r}_n| = |\bar{s}_n - \alpha| \leqq \sum_{r=1}^{M} |c_r| \, [\sigma(\lambda) + \varepsilon]^n, \quad n > n_0. \tag{4.4}$$

Let

$$\Lambda = \bigcup_j \Lambda_j. \tag{4.5}$$

Equation (4.4) gives immediately the following

Theorem 4. U is *effective* for S if $\sigma(\lambda) < 1$, $\lambda \in \Lambda$, *extensive* if this holds and S contains divergent sequences, *regular* for S if this holds and $S \subset X_C$, and accelerative for S if $S \subset X_C$ and

$$\sigma(\lambda) < |\lambda|, \quad \lambda \in \Lambda. \tag{4.6}$$

For an important class of methods, $\sigma(\lambda)$ is known:

Theorem 5. Let

$$P_n(\lambda) = p_n\left(\frac{2\lambda}{a} + 1\right) \Big/ \sigma_n(a), \quad a > 0 \tag{4.7}$$

where $\{p_m(x)\}$ is orthogonal on $[-1, 1]$ with respect to an integrable, positive not identically zero weight function $\omega(x)$, σ_m as in (3.5). Let the (Lebesgue) integrals

$$\int_{-\pi}^{\pi} f(\theta)\, d\theta, \qquad \int_{-\pi}^{\pi} |\ln f(\theta)|\, d\theta \tag{4.8}$$

exist, where

$$f(\theta) = \omega(\cos\theta)\, |\sin\theta|. \tag{4.9}$$

Then

$$\sigma(\lambda) = \left| \frac{\lambda^{\frac{1}{2}} + \sqrt{\lambda + a}}{\varrho} \right|^2, \quad \varrho = 1 + \sqrt{1 + a} > 2 \tag{4.10}$$

the branch cut of $z^{\frac{1}{2}}$ being taken along the negative real axis.

Proof. This follows directly from Szegö [9, Ch. 12].

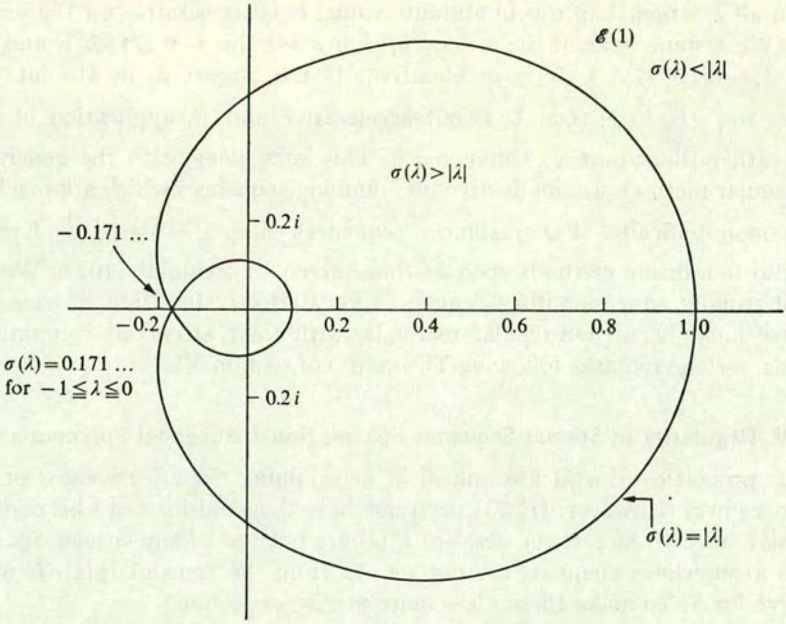
Next, we need some geometric concepts. Let $\mathscr{E}(a)$ be the exterior of the outer loop of the limacon type figure defined by

$$\lambda = \frac{a}{\sigma z(\varrho z - 2)}, \quad z = e^{i\theta}, \quad -\pi \leqq \theta \leqq \pi, \tag{4.11}$$

$$= \frac{a[\cos 2\theta - 2\cos\theta/\varrho] - i a [\sin 2\theta - 2\sin\theta/\varrho]}{\varrho^2 - 4\varrho\cos\theta + 4}. \tag{4.12}$$

This figure intersects the real axis at the points $(-a/\varrho^2,\ a/\varrho(\varrho+2),\ 1)$. The accompanying figure shows $\mathscr{E}$ for the important case $a = 1$.

Also let $\mathscr{F}(a)$ be the interior of the ellipse with center at $\left(-\frac{a}{2}, 0\right)$, major semi axis on the real axis of length $\frac{a}{2} + 1$, minor semi axis parallel to the imaginary axis of length $\sqrt{a+1}$.



We can now characterize the properties of U where $P_m(\lambda)$ is obtained from a set of polynomials orthogonal on $[-1, 1]$.

Theorem 6. Let $[\mu_{nk}]$ be obtained from a set of polynomials $p_m(x)$ orthogonal on $[-1, 1]$ with respect to a weight function satisfying (4.8) and (4.9) in accordance with

$$\begin{aligned} P_n(\lambda) &= p_n\left(\frac{2\lambda}{a}+1\right)\Big/\sigma_n(a) \\ &= \sum_{k=0}^{n} \mu_{n,k}\lambda^k \end{aligned} \tag{4.13}$$

and let S be as in (4.1). Then

i) if $S\subset X_C$, U is regular for S.

ii) U is effective for S if $\Lambda\subset\mathscr{F}(a)$.

iii) U is extensive for S if S contains divergent sequences (at least one $|\lambda_j|\geqq 1$) and $\Lambda\subset\mathscr{F}(a)$

iv) U is accelerative for S if $S\subset X_C$ and $\Lambda\subset\mathscr{E}(a)$.

Proof. Obvious.

Let's look at the important case where the λ_j in (4.1) are all real.

First, if $\lambda\in[-1, 0]$, by our square root convention

$$\sigma(\lambda)=\left(\frac{a}{\varrho}\right)^2=(\sqrt{a+1}-1)^2 \tag{4.14}$$

and for all λ_j larger than this in absolute value, U is accelerative for the sequence (4.1). (We assume none of the c_j's are 0). For $a=1$ this is 0.171 57 … and so for $\lambda_j \in (-1, -0.17157\ldots)$, U is accelerative. If the largest λ_j in absolute value belongs to $\left(-\left(\frac{a}{\varrho}\right)^2, 1\right)$ then U is not accelerative and an application of U will harm rather than improve convergence. This goes along with the general rule that regular methods are ineffective for summing sequences which approach their limits monotonically. For instance, sequences like $s_n = 1 + \left(\frac{1}{2}\right)^n$. Even the powerful non-linear methods such as those given by Schmidt [10] or Wimp [8] do not usually sum monotone sequences very effectively. Here it seems one's recourse must be to non-regular methods, with their attendent computational hazards, see the remarks following Theorem 9 of section VI.

V. Regularity in Special Sequence Spaces, Non-Orthogonal Polynomials

The properties of $\sigma(\lambda)$ are crucial in determining the effectiveness of U for sequences in S. However, $\{P_n(\lambda)\}$ need not be orthogonal for U to be regular or extensive in S. In fact, if the zeros of $P_n(\lambda)$ are not too widely spaced apart and if Λ is a set whose members are not too far from the zeros of $P_n(\lambda)$, U will be effective for S. To make these ideas more precise we define

$$P_n(\lambda) = Q_n(\lambda)/Q_n(1), \quad Q_n(\lambda) = \prod_{k=1}^{n} (\lambda - \lambda_{n,k}), \tag{5.1}$$

$$\mathscr{W} = \{\lambda_{n,k}\}, \quad k = 1, 2, \ldots, n = n_0, n_0+1, \ldots, n \geqq n_0, \tag{5.2}$$

$$\varkappa = \inf_{z \in \mathscr{W}} |z-1| \tag{5.3}$$

and

$$S_\gamma = \bigcup_{z \in \mathscr{W}} \{\lambda \mid |\lambda - z| < k\gamma, \quad \gamma < 1\}. \tag{5.4}$$

We now show

Theorem 7. For some n_0 let $\mathscr{W}$ be bounded, $\Lambda \subset S_\gamma$ for some $0<\gamma<1$, $\varkappa>0$, and S_γ be a connected set. Λ as in (4.5). Then

$$\sigma(\lambda) < \gamma, \quad \lambda \in \Lambda \tag{5.5}$$

and U is effective for all sequences in S.

Proof. Since S_γ is an open connected union of circles there exists a simple closed curve Γ within S_γ which contains both $\mathscr{W}$ and any $\lambda_0 \in \Lambda$. By a maximum modulus argument

$$|z - \lambda_{n,k}| < \varkappa\gamma, \quad z \in \Gamma, \quad n \geqq n_0. \tag{5.6}$$

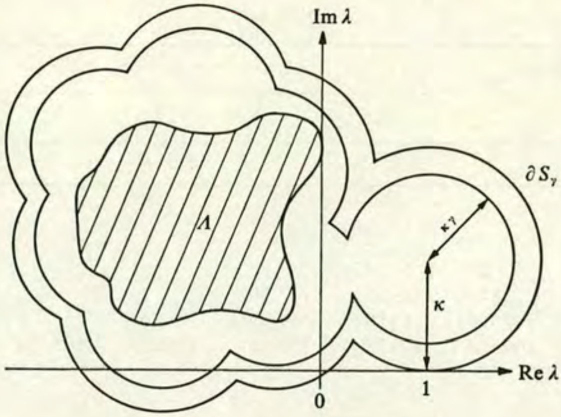
Since

$$P_n(\lambda_0) = \frac{1}{2\pi i} \int_\Gamma \frac{Q_n(z)\, dz}{Q_n(1)(z-\lambda_0)} \tag{5.7}$$

we have

$$|P_n(\lambda_0)| \leqq \frac{(\varkappa\gamma)^n L(\Gamma)}{2\pi\beta\varkappa^n} = M_0 \gamma^n, \tag{5.8}$$

$$\beta = \inf_{z \in \mathscr{W}} |z - \lambda_0| \tag{5.9}$$



an obvious modification being made if $\lambda_0 \in \mathscr{W}$. Thus the theorem follows. The theorem's geometric significance is that Λ must be contained in the union of circles around $\{\lambda_{n,k}\}$, $n \geqq n_0$, whose radius is the closest distance of the $\lambda_{n,k}$ to 1. A simple case where $\mathscr{W}$ is finite is illustrated below. The optimum value of γ is the smallest for which S_γ is connected and for which $\Lambda \subset S_\gamma$.

Theorem 7 yields a weaker result of Theorem 6 since in that case Theorem 7 says Λ must be in the region bounded below by $Im\,\lambda = -i$, above by $Im\,\lambda = i$, and on the left and right by semicircles opening out of radius 1 and centers $-a$, and 0 respectively. This region is interior to the known region of effectiveness, $\mathscr{F}(a)$.

Example 1. Here we choose for $P_n(\lambda)$ a class of polynomials having their zeros dense on a bounded curve in the complex plane. Let

$$P_n(\lambda) = \frac{\lambda^n {}_2F_0\left(-n, n+1 \,\middle|\, \dfrac{-1}{(2n+1)\lambda}\right)}{{}_2F_0\left(-n, n+1 \,\middle|\, \dfrac{-1}{2n+1}\right)} \tag{5.10}$$

$$= \lambda^{n+\frac{1}{2}} \frac{K_{n+\frac{1}{2}}((n+\frac{1}{2})\lambda)}{K_{n+\frac{1}{2}}(n+\frac{1}{2})} e^{(n+\frac{1}{2})(\lambda-1)}. \tag{5.11}$$

The $P_n(\lambda)$ are related to the Bessel polynomials. $P_n(\lambda)$ has its zeros asymptotically on a half-eye in the left half plane which intersects the negative real axis at $-0.66274\ldots$ and the imaginary axis at $\pm i$, and whose parametric equation is

$$\lambda = -\left(t^2 - t\tanh t\right)^{\frac{1}{2}} \pm i\,(t\coth t - t^2)^{\frac{1}{2}}, \quad 0 \leqq t \leqq t_0, \quad t_0 = 1.19968\ldots \tag{5.12}$$

see Olver [11, p. 352]. We give a short table of $[\mu_{n,k}]$.

Let

$$s_n = \lambda_1^n + \lambda_2^n, \quad \lambda_1 = \frac{-1+i}{2}, \quad \lambda_2 = \frac{-1-i}{2}. \tag{5.13}$$

Then $\{s_n\} = \{2, -1, 0, \frac{1}{2}, -\frac{1}{2}, \frac{1}{4}, 0, -\frac{1}{8}, \frac{1}{8}\ldots\}$ is a slowly converging null sequence, but because λ_1, λ_2 are close to the curve $\mathscr{W}$ as defined by (5.12), $\overline{s}_n$ converges rapidly, for example $\overline{s}_5 = -3.15 \times 10^{-3}$.

Table 4

n	Numerator $\mu_{n,k}$						Denominator $\mu_{n,k}$
	k						
	0	1	2	3	4	5	
0	1						1
1	2	3					5
2	12	30	25				67
3	120	420	588	343			1471
4	1680	7560	14580	14580	6561		44961
5	30240	166320	406560	559020	439230	161051	1802421

More precise information about the effect of U on members of S can be obtained by computing $\sigma(\lambda)$, which can be done by using Olver's results [11, p. 332].

If $|\arg\lambda| < \frac{\pi}{2}$,

$$\sigma(\lambda) = \frac{|e^{\lambda - \sqrt{1+\lambda^2}}(\sqrt{1+\lambda^2}+1)|}{\varrho_1}, \tag{5.14}$$

$\varrho_1 = (1+\sqrt{2})\,e^{1-\sqrt{2}}$ while if $|\arg\lambda| > \frac{\pi}{2}$ (i.e., λ is in the left half plane) $K_{n+\frac{1}{2}}$ is expressed as a linear combination of $I_{n+\frac{1}{2}}$ and $K_{n+\frac{1}{2}}$ of negative argument, see Erdélyi [12, ch. 7] and again Olver's results can be used. We have

$$\sigma(\lambda) = \frac{|\lambda e^{\lambda}|}{\varrho_2}\max\left[v(\lambda),\quad \frac{1}{v(\lambda)}\right],\quad \varrho_2 = e^{-1-\sqrt{2}}(1+\sqrt{2}), \tag{5.15}$$

$$v(\lambda) = \left|\frac{\lambda e^{\sqrt{1+\lambda^2}}}{1+\sqrt{1+\lambda^2}}\right|. \tag{5.16}$$

Example 2. Here $\mathscr{W}$ is the line segment $[-1, 0]$ and the zeros of $P_n(\lambda)$ are equally spaced. Let

$$\lambda_{n,k} = -\frac{k}{n},\qquad 1 \leqq k \leqq n, \tag{5.17}$$

$$P_n(\lambda) = \prod_{k=1}^{n}\left(\lambda + \frac{k}{n}\right)\Big/\left(1+\frac{k}{n}\right). \tag{5.18}$$

A simple computation shows

$$\sigma(\lambda) = \left|\frac{(\lambda+1)^{\lambda+1}}{4\lambda^{\lambda}}\right| \tag{5.19}$$

the branch cut of the function inside the absolute value signs being taken between $[-\alpha, 0]$. It is interesting to tabulate $\sigma(\lambda)$ for λ real.

Thus U can be a very efficient summation method if s_n is composed of terms which decay exponentially and alternate in sign. In fact, it will sum the divergent sequence λ^n if $-2 < \lambda \leqq 1$. Applied to the sequence $\sum_{k=0}^{n}(-1)^k/(k+1)$ they yield an error sequence $\bar{r}_1 = 5.7\times10^{-2}$, $\bar{r}_2 = 8.5\times10^{-4}$, $\bar{r}_3 = 6.0\times10^{-4}$, $\bar{r}_4 = 2.7\times10^{-5}$,

Table 5

λ	$\sigma(\lambda)$
-2	1
-1.8	0.861
-1.6	0.721
-1.4	0.578
-1.1	0.429
-1	0.349
-0.9	0.25
-0.8	0.181
-0.6	0.152

λ	$\sigma(\lambda)$
-0.5	0.128
-0.4	0.125
-0.2	0.128
-0.1	0.152
0	0.25
0.1	0.350
0.2	0.429
0.4	0.578

$\bar{r}_5 = 1.6 \times 10^{-5}$, $\bar{r}_6 = 8.8 \times 10^{-7}$. In general, U is about as effective as the Chebyshev weights, but, unfortunately, does not lend itself to the elegant two-dimensional formulation of the latter, see equations (3.13), (3.15).

Next, we examine $\sigma(\lambda)$ for the Romberg procedures described by Theorem 2 and

$$s_{n,m+1} = a_m s_{n+1,m} + (1 - a_m) s_{n,m}, \qquad n, m = 0, 1, 2, \ldots \tag{5.20}$$

where $\sum (a_k - 1)$ converges and $a_m \notin [0, 1]$. It is interesting that the above algorithm, effective as it is in certain situations, can never improve the exponential quality of convergence for exponential sequences, i.e., a convergent sequence like (4.1) will be transformed into an $s_{n,m}$ which converges at most as some power of m better as $m \to \infty$ than the original sequence. Also, sequences in S which diverge unboundedly will be mapped into divergent sequences. These remarks are a consequence of the following

Theorem 8. Let $a_m \to 1$, $a_m \neq 0$ and define $\mu_{m,k}$ for the above algorithm by

$$s_{n,m} = \sum_{k=0}^{m} \mu_{m,k} s_{n+k}. \tag{5.21}$$

Then

$$\sigma(\lambda) = \overline{\lim_{m\to\infty}} \, |P_m(\lambda)|^{\frac{1}{m}} = |\lambda|. \tag{5.22}$$

Proof. We have by (2.10)

$$P_m(\lambda) = \prod_{j=0}^{m-1} [a_j \lambda + (1 - a_j)] \tag{5.23}$$

so

$$P_{m+1}(\lambda) = \lambda P_m(\lambda) \left[\frac{1 + c_m(\lambda - 1)}{\lambda}\right], \tag{5.24}$$

$$c_m = a_m - 1,$$

an obvious modification in the argument being made if $\lambda = 0$. Applying Perron's theorem to the difference equation (5.24) yields the theorem, see Milne-Thomson [13, p. 548].

Of course, inferior convergence is the price that must be paid for an algorithm as simple and elegant as (5.20). Note that Theorem 8 provides in the sequence space S a generalization of Theorem 2, since we only require weakened hypotheses on a_m, i.e.,

Corollary. Let a_m satisfy the hypotheses of Theorem 8. Then $s_{n,m}$ is horizontally regular for convergent subsequences of S.

VI. Inverse Methods

We want to point out in closing that some interest attaches to the inverse U^* of Toeplitz methods, i.e.,

$$U^* = [\mu^*_{n,k}] \tag{6.1}$$

where

$$\bar{s}_n = \sum_{k=0}^{n} \mu_{n,k} s_k \tag{6.2}$$

if

$$s_n = \sum_{k=0}^{n} \mu^*_{n,k} \bar{s}_k \tag{6.3}$$

for all sequences $\{s_n\}$, $\{\bar{s}_n\}$.

Of course, the characteristic polynomial of U^* cannot usually be demonstrated. In one case, interestingly, this can be accomplished, viz., for the non-regular methods discussed in section I.

If

$$P_n(\lambda) = (-1)^n P_n^{(\tau-1,0)}(1-2\lambda) \tag{6.4}$$

we find from [12, v. 2, p. 212 (3)] that

$$P_n^*(\lambda) = \sum_{k=0}^{n} \mu^*_{n,k} \lambda^k = \frac{\left(\tau + 2\lambda \frac{d}{d\lambda}\right)}{(n+\tau)} V_n(\lambda) \tag{6.5}$$

where

$$V_n(\lambda) = F\left(\begin{matrix} -n, & 1 \\ n+\tau+1 & \end{matrix}\middle| -\lambda\right). \tag{6.6}$$

F denoting Gauss' function. Further,

$$\sigma^*(\lambda) = \lim_{n\to\infty} |P_n^*(\lambda)|^{\frac{1}{n}} = \begin{cases} 1, & |\lambda| < 1 \\ \max\left[1, \frac{|\lambda+1|^2}{4|\lambda|}\right], & |\lambda| > 1. \end{cases} \tag{6.7}$$

For $|\lambda| < 1$, (6.7) follows from (6.6) by dominated convergence and taking a termwise limit. For $|\lambda| > 1$, we consider the integral

$$\begin{aligned} I(\alpha, \beta; \lambda) &= \int_0^1 (1-t)^{n+\alpha}(1+\lambda t)^{n+\beta} dt \\ &= (1+\lambda)^{n+\beta} \int_0^1 z^{n+\alpha}(1-\gamma z)^{n+\beta} dz, \qquad \gamma = \frac{\lambda}{1+\lambda} \\ &= (1+\lambda)^{n+\beta} \left[\int_0^{1/\gamma} + \int_{1/\gamma}^{1}\right]. \\ &= (1+\lambda)^{2n+\alpha+\beta+1} \lambda^{-n-\alpha-1} B(n+\alpha+1, n+\beta+1) - 1/\lambda I(\beta, \alpha | 1/\lambda) \end{aligned} \tag{6.8}$$

The use of Stirling's formula above and the relation

$$(n+\alpha+1)\, I(\alpha,\beta|\lambda) = F\left(\begin{matrix}-n-\beta,\ 1\\ n+\alpha+2\end{matrix}\middle| -\lambda\right) \tag{6.9}$$

shows (6.7) for $|\lambda|>1$.

A quick computation shows that U^* is regular, but it is a poor method to use on exponential sequences, since if $s_n \in S$, $\bar{s}_n$ will converge more slowly than λ^n for all $|\lambda|<1$.

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EQUICONVERGENCE THEOREMS FOR SERIES WHOSE TERMS SATISFY A DIFFERENCE EQUATION*

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Abstract. We discuss the error of expansions of functions in series of function $\{p_n(z)\}$ which are defined by a generating function. If $p_n(z)$ satisfies a linear difference equation of a certain kind, then the error of the expansion may be simply related to the error of a much simpler Taylor series. Some of our formulas are of practical value in summing expansions in $p_n(z)$ with given coefficients, and we give several applications to expansions in Pollaczek polynomials $P_n^\lambda(x; a, b)$.

1. Introduction. In this paper we are concerned with the convergence properties of expansions in functions $\{p_n(z)\}$ which are defined by a generating function

$$K(z, w) = \sum_{n=0}^{\infty} p_n(z)w^n.$$

It is found that if $p_n(z)$ satisfies a linear recursion relationship of a rather general kind, the error of an expansion in these functions can be simply related to the error of a much simpler expansion, that is, a Taylor series with related coefficients. In the course of our analysis, formulas are given which are of practical value in "summing" the $p_n(z)$ series when the sum of the related Taylor series is known, and we present applications of our results to expansions in the so-called Pollaczek polynomials $P_n^\lambda(z; a, b)$. Also, certain neat statements can be made about the convergence of the $p_n(z)$ series when the related function is an entire function of exponential order.

2. Formulas. Let

$$K(z, w) = \sum_{n=0}^{\infty} p_n(z)w^n \tag{1}$$

for (z, w) belonging to some region of $C \times C$. In particular, we assume $z \in Z \subset C$ and for each fixed $z \in Z$, (1) has a nonzero radius of convergence. Also let

$$f(z) = \sum_{n=0}^{\infty} \mathscr{L}_n(f)p_n(z) \tag{2}$$

converge absolutely. We define

$$\Phi(z) = \sum_{n=0}^{\infty} \sigma_n \mathscr{L}_n(f)z^n, \qquad \sigma_n > 0. \tag{3}$$

(In general $\{\sigma_n\}$ is a sequence which will insure that (3) is analytic in some neighborhood of the origin.) Then

$$\sigma_k \mathscr{L}_k(f) = \frac{1}{2\pi i}\int_C \frac{\phi(t)}{t^{k+1}}\,dt, \tag{4}$$

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where ϕ is analytic on and within C, a simple closed contour encircling the origin.[1]

Let

$$K^*(z,t) = \frac{1}{t}\sum_{n=0}^{\infty}\frac{q_n(z)}{t^n}, \qquad q_n(z) = \frac{p_n(z)}{\sigma_n}. \tag{5}$$

If we assume that

$$\overline{\lim_{n\to\infty}} \ |q_n(z)|^{1/n} = \zeta(z) < \infty, \tag{6}$$

then (5) will converge if $|t| > \zeta(z)$, z fixed.

We then can write

$$\begin{aligned} E_n[f(z)] &= \frac{1}{2\pi i}\int_C \phi(t)K_n^*(z,t)\,dt, \\ E_n[f(z)] &= f(z) - \sum_{k=0}^{n-1}\mathscr{L}_k(f)p_k(z), \\ K_n^*(z,t) &= \frac{1}{t}\sum_{k=n}^{\infty}\frac{q_k(z)}{t^k} \end{aligned} \tag{7}$$

provided that on C, ϕ is analytic and $|t| > \zeta(z)$.

For $n = 0$ this becomes

$$f(z) = \frac{1}{2\pi i}\int_C \phi(t)K^*(z,t)\,dt. \tag{8}$$

Most practical interest attaches to the generating function (1) when $p_n(z)$ satisfies a recursion relationship

$$\sum_{\nu=0}^{r} A_\nu(n)p_{n+\nu}(z) = 0, \qquad n = 0, 1, 2, \cdots, \tag{9}$$

where not all the A_ν are zero. Often the $A_\nu(n)$ are rational functions of n, for example, if $p_n(z)$ is one of the classical orthogonal polynomials. When this happens is made clear by the following theorem.

THEOREM 1. *$p_n(z)$ satisfies* (9) *with $A_\nu(n)$ a rational function of n if and only if K satisfies the differential equation*

$$\sum_{\nu=0}^{s}\frac{\partial^\nu}{\partial w^\nu}K(z,w)B_\nu(w) = 0, \tag{10}$$

where the B_ν are polynomials in w not all of which are zero.

The proof of this theorem is just a matter of series manipulations. Since it depends on properties of generating functions so well described by other writers (see [2, vol. 3; 3]) we omit it. For instance, to show (10) implies (9), one would write

$$\frac{\partial^r}{\partial w^r}K = \sum_{n=0}^{\infty} p_{n+r}(z)(n+1)_r w^n,$$

substitute in (10), and rearrange. Note that A_ν, B_ν in general will depend on z.

[1] For a discussion of the theory of (1), (2), (3), see [1].

Now, with the exception of certain logarithmic cases, the consideration of which introduces only mechanical, not conceptual, difficulties into our analysis, the work of Birkhoff and Trjitzinsky [4], [5] shows that if p_n satisfies (9), then we can express it as a linear combination

$$p_n(z) = \sum_{h=1}^{r'} V_h(n), \qquad r' \leqq r, \tag{11}$$

where the $V_h(n)$ are functions of z which have the following asymptotic representations:

$$\begin{aligned} V_h(n) &\sim e^{Q_h(n)} s_h(n), \qquad n \to \infty, \\ Q_h(n) &= \mu_{0,h} n \ln n + \mu_{1,h} n + \mu_{2,h} n^{(\rho-1)/\rho} + \cdots + \mu_{\rho,h} n^{1/\rho}, \\ s_h(n) &= n^{\theta_h}[\alpha_{0,h} + \alpha_{1,h} n^{-1/\rho} + \cdots], \end{aligned} \tag{12}$$

where ρ is an integer $\geqq 1$, $\alpha_{0,h} \neq 0$ and $\mu_{0,h}$ is an integral multiple of $1/\rho$. (12) is a natural generalization of a Poincaré type asymptotic expansion. For the properties of such expansions, see the cited references.

We now assume that σ_n can also be chosen so that

$$q_n(z) = \sum_{h=1}^{r'} W_h(n), \tag{13}$$

where a representation like (12) holds for each $W_h(n)$ but $\mu_{0,h} = 0$, $1 \leqq h \leqq r'$.

In [6], it is shown that $K_n^*(z,t)$ can be written as a linear combination of functions $U_h(n,t)$,

$$K_n^*(z,t) = \sum_{h=1}^{r'} U_h(n,t), \tag{14}$$

where each U_h has the asymptotic expansion

$$\begin{aligned} U_h(n,t) &\sim \frac{e^{Q_h(n)}}{t^n} s_h^*(n,t), \qquad n \to \infty, \\ s_h^*(n,t) &= n^{\theta_h}[\beta_{0,h} + \beta_{1,h} n^{-1/\rho} + \cdots]. \end{aligned} \tag{15}$$

The leading constants in the latter series may be found by the method of undetermined coefficients explained in the above reference. We have, for example,

$$\begin{aligned} \beta_{0,h} &= \frac{\alpha_{0,h}}{t - w_h}, \\ \beta_{1,h} &= \frac{\alpha_{1,h}}{t - w_h} + \frac{w_h \mu_{2,h}(\rho - 1)\alpha_{o,h}}{\rho(t - w_h)^2} \qquad (\rho > 1) \\ &= \frac{\alpha_{1,h}}{t - w_h} + \frac{w_h \theta_h \alpha_{0,h}}{(t - w_h)^2} \qquad (\rho = 1), \end{aligned} \tag{16}$$

where $w_h = e^{\mu_{1,h}} \neq 0$ and where $\theta_h, \alpha_{j,h}, Q_h, \mu_{j,h}$ are the parameters corresponding to the decomposition (12).

Now from the development in [5] the $\beta_{j,h}$ are seen to be continuous functions (in fact, analytic) if $t \neq w_h$. An easy argument using Theorem 7.13 of Rudin [7] shows the asymptotic representation (15) holds uniformly on all compact t sets not containing w_h. Note also that w_h is strictly within C, since by convergence of (5) and (15),

$$\overline{\lim_{n\to\infty}} \left|\frac{q_n(z)}{t^n}\right|^{1/n} = \frac{\max_h |w_h|}{|t|} < 1, \qquad t \in C. \tag{17}$$

Thus the representation (15) holds uniformly on C.

Substituting (14) in (7) gives

$$E_n[f(z)] = \sum_{h=1}^{r'} Y_h(n), \tag{18}$$

$$Y_h(n) \sim e^{Q_h(n)} n^{\theta_h}[\gamma_{0,h} + \gamma_{1,h} n^{-1/\rho} + \cdots], \qquad n \to \infty. \tag{19}$$

Since the $\gamma_{j,h}$ above depend on n, (19) must be interpreted as

$$\begin{aligned} &e^{-Q_h(n)} n^{-\theta_h} Y_h(n) - \sum_{j=0}^{r} \gamma_{j,h} n^{-j/\rho} = O(\gamma_{r+1,h} n^{-(r+1)/\rho}), \\ &n \to \infty, \qquad r = 0, 1, 2, \cdots. \end{aligned} \tag{20}$$

We have

$$\gamma_{0,h} = \frac{\alpha_{0,h}}{2\pi i} \int_C \frac{\phi(t)\, dt}{t^n(t - w_h)}, \tag{21}$$

$$\begin{aligned} \gamma_{1,h} &= \frac{1}{2\pi i} \int_C \frac{\phi(t)}{t^n} \left(\frac{\alpha_{1,h}}{t - w_h} + \frac{\lambda}{(t - w_h)^2}\right) dt, \\ \lambda &= w_h \mu_{2,h} \alpha_{0,h}(\rho - 1)/\rho \qquad (\rho > 1) \\ &= w_h \theta_h \alpha_{0,h} \quad (\rho = 1). \end{aligned} \tag{22}$$

Since ϕ is analytic on and within C we have

$$\begin{aligned} \frac{R_n[\phi, \alpha]}{\alpha^n} &= \frac{1}{2\pi i} \int_C \frac{\phi(t)\, dt}{t^n(t - \alpha)}, \\ R_n[\phi, \alpha] &= \phi(\alpha) - \sum_{k=0}^{n-1} \sigma_k \mathscr{L}_k(f) \alpha^k. \end{aligned} \tag{23}$$

Also,

$$\frac{R_{n-1}[\phi', \alpha]}{\alpha^n} - \frac{n R_n[\phi, \alpha]}{\alpha^{n+1}} = \frac{1}{2\pi i} \int_C \frac{\phi(t)\, dt}{t^n(t - \alpha)^2}. \tag{24}$$

We find that

$$\gamma_{0,h} = \frac{\alpha_{0,h}}{w_h^n} R_n[\phi, w_h], \tag{25}$$

$$\gamma_{1,h} = \frac{(\alpha_{1,h} - n\lambda)}{w_h^n} R_n[\phi, w_h] + \frac{\lambda}{w_h^n} R_{n-1}[\phi', w_h]. \tag{26}$$

THEOREM 2. *Let $\phi(t)K_n^*(z, t)$, where ϕ, K_n^* are given by* (3) *and* (5), *be analytic in some annulus $A = \{t | r_1 < |t| < r_2\}$. Let $q_n(z)$ have the decomposition* (13), *where*

$$W_h(n) \sim e^{Q_h(n)} s_h(n), \qquad n \to \infty, \tag{27}$$

$\mu_{0,h} = 0, s_h, Q_h$ *as in* (12). *Then $E_n[f(z)]$ has the asymptotic representation* (18), (19) *with leading coefficients $\gamma_{0,h}$, $\gamma_{1,h}$ related to the error of the Taylor series for $\phi(t)$ by* (25), (26).

Note r_1 is the singularity of largest modulus of K_n^*, r_2 the singularity of smallest modulus of ϕ. If $r_1 < r_2$ then a path C can be determined so the analysis above holds. Also, the fact that (1) has a nonzero radius of convergence is not essential to the analysis, only that the series for K_n^* converge. Thus the p_n may be generated by a "formal" power series.

Let C be a circle of radius R. Then

$$\left| \frac{R_n[\phi, \alpha]}{\alpha^n} \right| \leqq M_\phi(R)/R^{n-1}(R - |\alpha|), \tag{28}$$

where $M_\phi(R)$ is the maximum of $|\phi(t)|$ on $t \in C$. It follows that there exist constants A_h such that

$$|Y_h(n)| < \frac{e^{\operatorname{Re} Q_h(n)} n^{\operatorname{Re} \theta_h}}{R^{n-1}(R - |w_h|)} |\alpha_{0,h}| M_\phi(R) \left| 1 + \frac{A_h}{n^{1/\rho}} \right|, \qquad n > n_0, \tag{29}$$

and this can be a useful upper bound in (18). It is interesting to see what happens when ϕ is entire. Let

$$M_\phi(R) = O(e^{R^{\sigma+\varepsilon}}), \qquad R \to \infty \qquad \text{for all } \varepsilon > 0. \tag{30}$$

Then the function

$$g_n(R) = e^{R^{\sigma+\varepsilon}}/R^n$$

occurs in (27). This function attains a minimum at

$$R_0 = \left(\frac{n}{\sigma + \varepsilon} \right)^{1/(\sigma+\varepsilon)} \tag{31}$$

and Stirling's formula shows that

$$g_n(R_0) = \frac{Cn^{-1/2}}{\Gamma(n/(\sigma + \varepsilon))} [1 + O(n^{-1})]. \tag{32}$$

We have shown the following result.

THEOREM 3. *Let $\phi(t)$ be an entire function of order σ and let the hypotheses of Theorem* 2 *hold. Then for every $\varepsilon > 0$,*

$$E_n[f(z)] = \Gamma\left(\frac{n}{\sigma + \varepsilon} \right)^{-1} \sum_{h=1}^{r} O_h[n^{\operatorname{Re} \theta_h - 1/2} e^{\operatorname{Re} Q_h(n)}], \qquad n \to \infty. \tag{33}$$

The order terms in the sum depend on ε.

3. Applications. As an example of the application of some of the previous formulas, we will consider the generating function for the Pollaczek polynomials,

$P_n^\lambda(z; a, b)$, [1, vol. 2, p. 218]. We will take $\sigma_n \equiv 1$.

$$K(z, w) = \left(1 - \frac{w}{w_1}\right)^{-\lambda + iA}\left(1 - \frac{w}{w_2}\right)^{-\lambda - iA} = \sum_{n=0}^{\infty} P_n^\lambda(z; a, b) w^n, \tag{34}$$

where

$$\begin{aligned} w_1 &= z + i\sqrt{1 - z^2}, \\ w_2 &= \frac{1}{w_1} = z - i\sqrt{1 - z^2}, \\ A &= -(az + b)/\sqrt{1 - z^2}, \qquad a, b, \lambda \text{ real}. \end{aligned} \tag{35}$$

Here we mean

$$\begin{aligned} &\sqrt{1 - z^2} = |1 - z^2|^{1/2} e^{(i/2)(\phi_1 + \phi_2 - \pi)}, \\ &\phi_1 = \arg(z - 1), \quad \phi_2 = \arg(z + 1), \qquad -\pi < \phi_1 \leqq \pi, \quad 0 \leqq \phi_2 < 2\pi. \end{aligned} \tag{36}$$

Darboux's method may be applied to (34) to obtain an asymptotic representation for $p_n(z)$.

We get $r' = 2$,

$$\begin{aligned} &p_n(z) = P_n^\lambda(z; a, b) = V_1(n) + V_2(n), \\ &V_1(n) \sim \frac{(1 - w_1/w_2)^{-\lambda - iA}}{\Gamma(\lambda - iA)} w_2^n n^{\lambda - 1 - iA}\left[1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \cdots\right], \\ &V_2(n) \sim \frac{(1 - w_2/w_1)^{-\lambda + iA}}{\Gamma(\lambda + iA)} w_1^n n^{\lambda - 1 + iA}\left[1 + \frac{d_1}{n} + \frac{d_2}{n^2} + \cdots\right], \qquad n \to \infty. \end{aligned} \tag{37}$$

This reveals the remarkable fact that the powers of n in the algebraic portion of the asymptotic expansion of the Pollaczek polynomials *depend on* z, viz., $\theta_1 = \lambda - 1 - iA$, $\theta_2 = \lambda - 1 + iA$. For none of the classical orthogonal polynomials is this the case. (The Gegenbauer polynomials result when $a = b = 0$ so for these polynomials, $\theta_1 = \theta_2 = \lambda - 1$.)

For these Pollaczek polynomials we further have $\rho = 1, \mu_{0,1} = \mu_{0,2} = 0$,

$$e^{\mu_{1,h}} = w_2, \quad e^{\mu_{2,h}} = w_1$$

and $\alpha_{0,h}$ may be read off (37).

K as a function of w has branch cuts at w_1 and w_2. We assume a branch cut is established between these two points to make K single-valued. Now consider

$$\phi(z) = \left(1 - \frac{z}{\gamma}\right)^\mu, \qquad |\gamma| > \max(|w_1|, |w_2|). \tag{38}$$

We have

$$K^*(z, t) = \frac{1}{t}\left(1 - \frac{w_2}{t}\right)^{-\lambda + iA}\left(1 - \frac{w_1}{t}\right)^{-\lambda - iA}, \tag{39}$$

$$\alpha_1 = -\lambda + iA, \quad \alpha_2 = -\lambda - iA,$$

so

$$f(z) = \frac{1}{2\pi i}\int_C \frac{(1 - t/\gamma)^\mu}{t}\left(1 - \frac{w_2}{t}\right)^{-\lambda + iA}\left(1 - \frac{w_1}{t}\right)^{-\lambda - iA} dt \tag{40}$$

and C is a circle on which

$$\max(|w_1|, |w_2|) < |t| < |\gamma|,$$

a branch cut having been made from γ to $\infty e^{i\arg\gamma}$.

Making the change of variable $w_1 u + (1 - u)w_2 = t$ in the integral and expanding $t^{2\lambda - 1}, (1 - t/\gamma)^\mu$ gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}(w_1^2 - 1)^{1-2\lambda}\left(1 - \frac{w_2}{\gamma}\right)^\mu \sum_{m,n=0}^{\infty} \frac{(-\mu)_m(-1)^m(1 - 2\lambda)_n}{m!n!w_2^n(\gamma - w_2)^m} \\ &\quad \cdot (w_2 - w_1)^{m+n}\int_{C'} u^{\alpha_1 + m + n}(u - 1)^{\alpha_2}\, du, \end{aligned} \tag{41}$$

where C' is a simple closed curve around $[0, 1]$ in the clockwise direction. But this is a known integral for the beta function [1, vol. 1, p. 15]. We thus have obtained the expansion

$$\begin{aligned} &(w_1^2 - 1)^{1-2\lambda}\left(1 - \frac{w_2}{\gamma}\right)^\mu \frac{\Gamma(1 - \lambda + iA)\Gamma(1 - \lambda - iA)}{\Gamma(2 - 2\lambda)\pi}\sin[\pi(1 - \lambda - iA)] \\ &\cdot F_1\left(1 - \lambda + iA, -\mu, 1 - 2\lambda, 2 - 2\lambda; \frac{1 - w_1^2}{1 - \gamma w_1}, 1 - w_1^2\right) \\ &= \sum_{n=0}^{\infty} P_n^\lambda(z; a, b)\frac{(-\mu)_n\gamma^{-n}}{n!}. \end{aligned} \tag{42}$$

The function F_1 is Appell's hypergeometric function [2, vol. 1, p. 224 (6)].

If instead we start with the function

$$\phi(z) = e^{\xi z}, \tag{43}$$

a similar analysis yields the expansion

$$f(z) = \Phi_2(\lambda - iA, \lambda + iA, 1, w_2\xi, w_1\xi) = \sum_{n=0}^{\infty}\frac{\xi^n}{n!}P_n^\lambda(z; a, b). \tag{44}$$

Now for t fixed,

$$R_n[\phi, t] = R_n[e^{\xi t}] = \frac{(\xi t)^n}{n!}[1 + O(n^{-1})], \tag{45}$$

so

$$\gamma_{0,h} = \frac{\alpha_{0,h}\xi^n}{n!} \tag{46}$$

and so

$$E_n[f(z)] = Y_1(n) + Y_2(n), \tag{47}$$

$$\begin{aligned} Y_1(n) &\sim \frac{(\xi w_2)^n}{n!} n^{\lambda-1-iA} \alpha_{0,1} \left[1 + \frac{a_1}{n} + \cdots \right], \qquad n \to \infty, \\ Y_2(n) &\sim \frac{(\xi w_1)^n}{n!} n^{\lambda-1+iA} \alpha_{0,2} \left[1 + \frac{b_1}{n} + \cdots \right], \qquad n \to \infty. \end{aligned} \tag{48}$$

Roughly speaking, this means that $E_n[f(z)]$ behaves like $\xi^n P_n^\lambda(z; a, b)/n!$ as $n \to \infty$. Since in this case $\mathscr{L}_n(f)$ can be estimated asymptotically by Stirling's formula, the theory in [3] or a straightforward majorization argument based on the estimates (37) could be used to obtain an asymptotic formula for $E_n[f(z)]$. However, $\mathscr{L}_n(f)$ will not in general be so tractable.

Similar formulas pertaining to the expansion (42) can also be readily deduced.

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expected that these smaller systems will be used for data entry devices in conjunction with some degree of editing. However, the management of large files is a complex and sophisticated process which is inherently slow so that there is no great advantage to these smaller machines for this purpose.

It is to be expected that computers will begin to communicate with each other to a much greater extent than now. The communication which now exists is that of a master/slave relationship in which data is transferred to the control facility using line disciplines controlled by this central facility. However, it is to be expected that computers will communicate with each other on a "need to know" basis; that is, if information is called for by a small machine it should be in position to transmit a request for a record to another machine elsewhere.

The effect of these two developments will be to remove the need for centralized large general-purpose computers, except to manage data files, and to replace them with a decentralized network. The management of this network will still be a centralized process as will the development of the software for the network.

The major application of such networks will be to access specialized data files, such as demographic, economic, or scientific data. These files will be maintained by using efficient access methods designed especially for them.

The time frame in which this will occur depends to a great extent upon the economics involved and to the control executed by the manufacturers over the basic patents. Experimental nets are operating now in a halting manner. By 1984 it would not be unreasonable to see commercial networks for large innovative commercial operations which have a significant amount of sophistication.

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ACCELERATION METHODS

INTRODUCTION

Let X be a metric space with metric d (see Simmons [1]) and suppose we have a sequence $\{s_n\}$ in X with a limit α in X, in other words,

$$\lim_{n\to\infty} d(s_n, \alpha) = 0 \tag{1}$$

Let T be a transformation mapping $\{s_n\}$ into another sequence $\{\bar{s}_n\}$

$$T\colon \{s_n\} \to \{\bar{s}_n\} \tag{2}$$

whose members also belong to X.

The purpose of this article is to discuss the many different kinds of transformations which have proved to be of value in the mathematics of computation. Of particular interest are those transformations which map $\{s_n\}$ into a sequence which converges to α more rapidly than the original sequence. Such transformations are called acceleration methods, see property 4 below.

Depending on the problem, X might be the space of real or complex numbers with $d(x, y) = | x - y |$, the space of $k \times 1$ complex vectors with

$$\alpha(x, y) = || x - y || \tag{3}$$

where $|| \cdot ||$ is the vector norm or the space of all $k \times k$ matrices with complex entries with

$$d(A, B) = || A - B || \tag{4}$$

where

$$|| A || = \max_{||x||=1} || Ax || \tag{5}$$

is the operator norm induced by the given vector norm. All these spaces are finite dimensional, but in many practical cases X is infinite dimensional. It might, for instance, be a Banach space, or even the space of all bounded linear transformations mapping one Banach space into another. In the latter case, the metric is the operator norm above (with max replaced by sup).

Our primary interest will be in scalar sequences which are treated in the following two sections. We will then give a brief description of acceleration methods for sequences in more general metric spaces.

There are several properties the transformation T might have which are important.

1. If T maps every convergent sequence $\{s_n\}$ in X into a sequence $\{\bar{s}_n\}$ which converges to the same limit, T is said to be *totally regular*.

2. If T maps every convergent sequence $\{s_n\}$ whose members belong to some space S of sequences in X into convergent sequences, then T is said to be *regular for the sequence space S*.

3. If T maps some divergent sequences into convergent sequences, then T is called *extensive* (for that sequence or class of sequences).

4. If T has the property that it is regular in the sequence space S and

$$\lim_{n\to\infty} \frac{d(\bar{s}_n, \alpha)}{d(s_n, \alpha)} = 0 \tag{6}$$

then T is called an *acceleration method* (or accelerative) for sequences in S. This means that the transformed sequence converges more rapidly than the original sequences. In this article, we will be concerned exclusively with transformations that are acceleration methods, at least for important classes of sequences or for the particular sequence under consideration.

5. If, in addition, X is a vector space with metric as in Eq. (3), we say that T is *linear* if

$$T(\alpha\{s_n\} + \beta\{t_n\}) = \alpha T\{s_n\} + \beta T\{t_n\} = \alpha\{\bar{s}_n\} + \beta\{\bar{t}_n\}$$

Otherwise, T is said to be nonlinear.

From here on, we will use the abbreviated notation $T(s_n) = \bar{s}_n$ even though $\bar{s}_n$ may depend on members of the sequence $\{s_n\}$ other than s_n alone. This should not cause confusion to the reader, since usually an explicit formula for $\bar{s}_n$ will be close by.

One important use of acceleration methods is in determining the solution of equations of the form

$$\psi(z) = 0 \tag{7}$$

This equation can be rewritten (in an infinite number of ways) in the form

$$z = F(z) \tag{8}$$

Now define the sequence s_n by taking s_0 arbitrary and

$$s_{n+1} = F(s_n), \qquad n = 0, 1, 2, \ldots \tag{9}$$

If s_n converges, it is clear that it converges to a root of Eq. (8). However, in practice this sequence usually converges slowly if it converges at all. What one then attempts to do is to determine an acceleration method which maps s_n into a sequence which converges rapidly enough to be useful computationally. Whether it converges to a root of Eq. (8) may then be determined by checking.

SCALAR ALGORITHM: LINEAR CASE

In this section we discuss transformations of scalar sequences $\{s_n\}$, i.e., the cases where the members of the sequence are real or complex numbers. Note that if X is any finite-dimensional vector space and $\{s_n\}$ is a sequence of vectors in X, then we may write

$$s_n = \sum_{j=1}^{M} a_{n,j} e_j \tag{10}$$

where M is the dimension of the space, $e_1, e_2, \ldots, e_M$ are a basis for the space, and the $a_{n,j}$ are scalars. If $s_n \to \alpha$ and

$$\alpha = \sum_{j=1}^{M} \alpha_j e_j \tag{11}$$

then each of the transformations T developed in the next few sections can be related to a transformation $T^*(s_n) = \bar{s}_n$ by

$$\bar{s}_n = \sum_{j=1}^{M} T(a_{n,j}) e_j \tag{12}$$

However there are many transformations of interest where the jth component of $\bar{s}_n$ depends on the jth component of s_n as well as other components of s_n. We will discuss these more general transformations in the final section of this article.

Throughout the next two sections we will use the notation

$$\Delta s_n = s_{n+1} - s_n, \qquad \Delta^2 s_n = \Delta(\Delta s_n), \text{ etc.}$$
$$r_n = s_n - \alpha \quad \text{if } s_n \to \alpha, \qquad \bar{r}_n = \bar{s}_n - \alpha \quad \text{if } \bar{s}_n \to \alpha \tag{13}$$

and we will often associate with the infinite series $\sum_{k=0}^{\infty} a_k$ the partial sum

$$s_n = \sum_{k=0}^{n} a_k \tag{14}$$

We will also encounter frequently the following "test" sequence, which will be used to judge the effectiveness of acceleration methods:

$$v_n \equiv v_n(\alpha, \beta, \lambda) = \sum_{k=0}^{n} \frac{\lambda^k}{(k+\beta)^\alpha}, \qquad \text{Re } \beta > 0 \tag{15}$$

Toeplitz Summability

Consider the following infinite array.

$$U = [\mu_{n,k}] = \begin{bmatrix} \mu_{00} & 0 & 0 & 0 \cdots \\ \mu_{10} & \mu_{11} & 0 & 0 \cdots \\ \mu_{20} & \mu_{21} & \mu_{22} & 0 \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix} \tag{16}$$

where

$$\sum_{k=0}^{n} \mu_{n,k} = 1 \tag{17}$$

U is called a *Toeplitz matrix* and the transformation

$$T(s_n) = \bar{s}_n = \sum_{k=0}^{n} \mu_{n,k} s_k, \qquad n = 0, 1, 2, \ldots \tag{18}$$

is called a *Toeplitz transformation corresponding to U.*

The following result is proved in Knopp [2].

If $\sum_{k=0}^{n} |\mu_{n,k}| < A$, *A independent of n, and* $\lim_{n\to\infty} \mu_{n,k} = 0$ *for each fixed k, then T is totally regular.*

Classic examples of regular Toeplitz summation matrices are given by the Cesaro weights

$$\mu_{n,k} = \frac{1}{n+1} \tag{19}$$

and the binomial weights

$$u_{n,k} = \binom{n}{k} 2^{-n} \tag{20}$$

TABLE 1

$$P_n(\lambda) = \frac{Q_n(\lambda)}{Q_n(1)} = \sum_{k=0}^{n} \mu_{n,k}\lambda^k$$

$Q_n(\lambda)$	Designation
$1 + \lambda + \lambda^2 + \cdots + \lambda^n$	Cesaro $\left(\mu_{n,k} = \dfrac{1}{n+1}\right)$
$(1 + \lambda)^n$	Binomial $\left(\mu_{n,k} = \binom{n}{k} \Big/ 2^n\right)$
$\sum_{k=0}^{n} \dfrac{(-n)_k(-1)^k(n)_k\lambda^k}{(\frac{1}{2})_k} = \cos[n \arccos(2\lambda + 1)]$	Chebyshev
$\prod_{k=0}^{n-1} \left(\lambda - \dfrac{1}{\sigma^k}\right), \quad \sigma > 1$	Romberg

It is easily shown these weights satisfy the conditions of the preceding theorem. For other examples, see Knopp [2] and Hardy [3].

To investigate the problem of when U is an acceleration method, we define a class of sequences S of the form

$$s_n = \alpha + \sum_{r=1}^{m} c_r\lambda_r^{\,n}, \qquad |\lambda_1| > |\lambda_2| > \cdots > |\lambda_m| \tag{21}$$

$c_r \neq 0$, $\lambda_r \neq 0, 1$. Our motivation for doing this is that many sequences encountered in practice can be expressed at least approximately as a constant plus a linear combination of "transients," exponentially increasing or decreasing perturbations of the form $\lambda_r^{\,n}$. Now let

$$P_n(\lambda) = \sum_{k=0}^{n} \mu_{n,k}\lambda^k \tag{22}$$

$$\sigma(\lambda) = \overline{\lim_{n\to\infty}}\, |P_n(\lambda)|^{1/n} \tag{23}$$

the latter being a pointwise limit. [Table 1 gives $P_n(\lambda)$ for some important totally regular Toeplitz methods.]

Then for some $\varepsilon > 0$ and n_0,

$$|\bar{s}_n - \alpha| < \sum_{r=1}^{m} |c_r| \,[\sigma(\lambda_r) + \varepsilon]^n, \qquad n > n_0 \tag{24}$$

Hence U will be *extensive* for these sequences for which

$$|\lambda_1| \geq 1, \qquad \sigma(\lambda_r) < 1, \qquad r = 1, 2, \ldots, m \tag{25}$$

and *accelerative* if

$$|\lambda_1| < 1, \qquad \sigma(\lambda_r) < |\lambda_1|, \qquad r = 1, 2, \ldots, m \tag{26}$$

For the binomial weights in Table 1, $P_n(\lambda) = (1 + \lambda)^n/2^n$, $\sigma(\lambda) = | 1 + \lambda | /2$, so T is extensive if $| \lambda_1 | \geq 1$, $| 1 + \lambda_r | < 2$ and accelerative if $| \lambda_1 | < 1$, $| 1 + \lambda_r | < 2 | \lambda_1 |$. For instance, T sums (to zero) $(-2)^n$ and makes $(-1/2)^n$ converge more rapidly. It will sum every sequence of the form of Eq. (21), convergent or not, provided the λ_r all lie in the circle with center at -1, radius 2.

The Chebyshev weights in Table 1 are obtained by taking $(-1)^k \mu_{n,k}$ to be proportional to the coefficient of x^k in the polynomial

$$C_n(x) = \cos[n \arccos(2x - 1)] \tag{27}$$

$\mu_{n,k}$ is then determined by making $\Sigma \mu_{n,k} = 1$. For the properties of the above polynomials, see Lanczos [4]. The motivation for choosing such a set of weights is as follows. Let $\{p_n(x)\}$ be a set of polynomials orthogonal with respect to the weight function $w(x)$ over the interval $[-1, 1]$, i.e.,

$$\int_{-1}^{1} w(x) p_n(x) p_m(x) = \begin{cases} 0, & m \neq n \\ h_n \neq 0, & m = n \end{cases} \tag{28}$$

We assume $w(x)$ is positive, integrable, and not identically zero.

Now consider the polynomials

$$p^*_n(x) = p_n\left(\frac{2x}{a} + 1\right) = \sum_{k=0}^{n} a_{n,k} x^k \tag{29}$$

which are orthogonal over the interval $[-a, 0]$ with respect to the weight function $w[(2x/a) + 1]$. If $w(x)$ satisfies certain integrability conditions (see Szegö [5, ch. 12]) it can be shown that

$$\lim_{n\to\infty} \left| \frac{p^*_n(x)}{p^*_n(1)} \right|^{1/n} = \left| \frac{\sqrt{x} + \sqrt{a + x}}{1 + \sqrt{a + 1}} \right|^2 \tag{30}$$

We will now examine the Toeplitz matrix where

$$\mu_{n,k} = a_{n,k} \Big/ \sum_{k=0}^{n} a_{n,k} \tag{31}$$

so

$$\sigma(\lambda) = \left| \frac{\sqrt{\lambda} + \sqrt{a + \lambda}}{1 + \sqrt{a + 1}} \right| \tag{32}$$

Thus $\sigma(\lambda) < \rho$ for all values of ρ for which

$$| \sqrt{\lambda} + \sqrt{a + \lambda} | < (1 + \sqrt{a + 1})\rho^{\frac{1}{2}} \tag{33}$$

This inequality is satisfied for all points $\lambda \in D_R$, where D_R is the interior of an ellipse, center at $-a/2$, major axis coinciding with the real axis, major semi-axis of length $(a/4)[R + (1/R)]$, minor semi-axis of length $(a/4)[R - (1/R)]$, and

$$R = \frac{\rho}{a}(1 + \sqrt{a + 1})^2 \tag{34}$$

Note that for all positive ρ

$$\frac{a}{4}\left(R+\frac{1}{R}\right)>\frac{a}{2} \tag{35}$$

Thus D_R is as in Fig. 1.

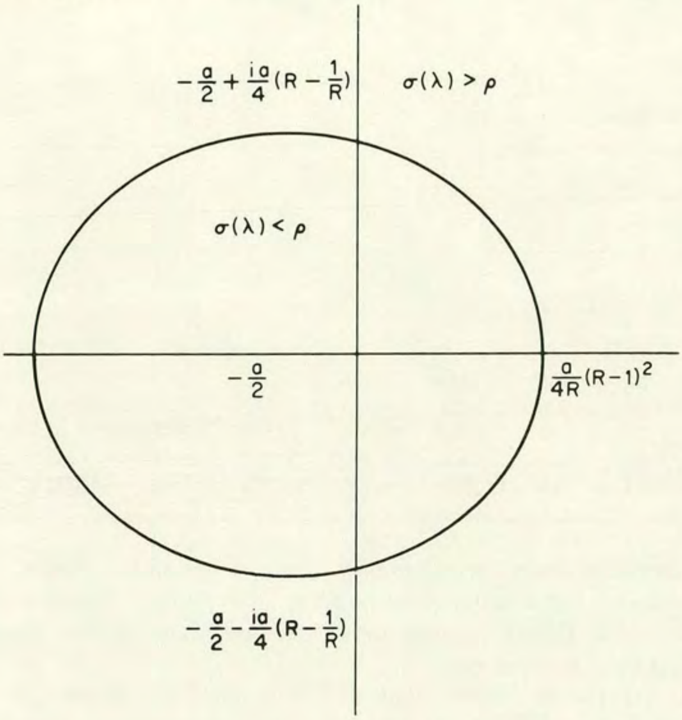

Figure 1.

The Chebyshev weights correspond to the choice $p_n(x) = T_n{}^*(x)$, $a = 1$. Then $R = \rho(3 + 2\sqrt{2})$. In general we can say for a Toeplitz matrix whose entries are given by Eq. (31) that:

1. T is *regular* for all convergent sequences of the form (21).
2. T is *extensive* if $|\lambda_1| \geq 1$ and $\lambda_j \in D_1$.
3. T is *accelerative* if $|\lambda_1| < 1$ and $\lambda_j \in D|\lambda_1|_{|\lambda_1|}$.
4. T sums all Sequences (21), convergent or not, if $\lambda_j \in D_1$.

This is the ellipse with center at $-(a/2)$, major axis the interval $[-(a/2) - 1, 1]$, and minor semi-axis of length $\sqrt{a+1}$.

For the Chebyshev weights $a = 1$ and

$$\mu_{n,k} = a_{n,k} \Big/ \sum_{k=0}^{n} a_{n,k} \tag{36}$$

$$a_{n,k} = \binom{n}{k} \frac{(n)_k}{(\frac{1}{2})_k} \tag{37}$$

Furthermore, it may be shown that if $P_n(\lambda)$ is *any* sequence of polynomials whose zeros are all negative and bounded, $P_n(1) = 1$, and

$$P_n(\lambda) = \sum_{k=0}^{n} \mu_{n,k} \lambda^k \tag{38}$$

TABLE 2

Chebyshev Weights, $\mu_{n,k} = a_{n,k}/w_n$

	$a_{n,k}$									
n/k	0	1	2	3	4	5	6	7	8	w_n
0	1									1
1	1	2								3
2	1	8	8							17
3	1	18	48	32						99
4	1	32	160	256	128					577
5	1	50	400	1120	1280	512				3363
6	1	72	840	3584	6912	6144	2048			19601
7	1	98	1568	9408	26880	39424	28672	8192		114243
8	1	128	2688	21504	84480	180224	212992	131072	32768	665857

then the transformation corresponding to $[\mu_{n,k}]$ is totally regular. Consequently, Chebyshev and other weights corresponding to systems of polynomials orthogonal on $[-a, 0]$ define totally regular transformations since all the zeros of the polynomial $P_n(\lambda)$ lie in this interval.

Practical experience dictates that if T is a good transformation for summing sequences of the form of Eq. (21), then it is a good general summation method. In any

TABLE 3

Salzer Weights, $\gamma = 1$ $\mu_{n,k} = a_{n,k}/w_n$

	$a_{n,k}$									
n/k	0	1	2	3	4	5	6	7	8	w_n
0	1									1
1	1	−2								−1
2	1	−8	9							2
3	1	−24	81	−64						−6
4	1	−64	486	−1024	625					24
5	1	−160	2430	−10240	15625	−7776				120
6	1	−384	10935	−81920	234375	−279936	117649			720
7	1	−896	45927	−573440	2734375	−5878656	8764801	−2097152		−5040
8	1	−2048	183708	−3670016	27343750	−94058496	161414428	−134217728	43046721	40320

particular case, a precise error bound for $[\bar{s}_n - \alpha]$ may be impossible to compute because our knowledge of the behavior of r_n is quite limited, and here is where the insight born of experience is the most valuable asset of the applied mathematician. Many times one suspects that T is summing the given sequence efficiently because $\bar{s}_{n+1}$ and $\bar{s}_n$ agree to more and more significant figures as n increases, and many times such a suspicion is justified.

Chebyshev and Salzer weights are given in Tables 2 and 3, respectively. It is our experience that the Chebyshev weights are the most useful ones for practical computations. Salzer weights are discussed in the section entitled Nonregular Methods.

Rational Approximations

The method of Toeplitz summation can be used to obtain rational approximations to functions. In many cases the rational approximations are extremely effective methods of computing the function which make excellent computer subroutines. We suppose, to illustrate our ideas, that the function to be approximated f is analytic at $z = 0$. Let $s_n(z)$ be the partial sum of its Taylor series

$$s_n(z) = \sum_{k=0}^{n} \frac{f^{(k)}(0)z^k}{k!} \tag{39}$$

We let

$$\mu_{n,k} = a_{n,k} \bigg/ \sum_{k=0}^{n} a_{n,k} \tag{40}$$

where

$$a_{n,k} = d_{n,k} z^{-k} \tag{41}$$

Then

$$\bar{s}_n(z) = \frac{z^n \sum_{k=0}^{n} d_{n,k} z^{-k} s_k(z)}{z^n \sum_{k=0}^{n} d_{n,k} z^{-k}} \tag{42}$$

There are a number of ways of choosing the $d_{n,k}$; see the volumes of Luke [6].

As an example, let $f(z) = e^z$,

$$d_{n,k} = (n+1)_k (-1)^k \binom{n}{k} \tag{43}$$

Then after some algebraic manipulation, it is found that $\bar{s}_n$ may be written

$$\bar{s}_n(z) = \frac{Q_n(-z)}{Q_n(z)}$$

$$Q_n(z) = (-z)^n \sum_{k=0}^{n} \binom{n}{k} (n+1)_k \left(-\frac{1}{z}\right)^k \tag{44}$$

for instance

$$s_0 = 1, \qquad s_1 = \frac{2+z}{2-z}$$

$$s_2 = \frac{12 + 6z + z^2}{12 - 6z + z^2}$$

$$s_3 = \frac{120 + 60z + 12z^2 + z^3}{120 - 60z + 12z^2 - z^3}, \text{ etc.} \tag{45}$$

It turns out that with the choice of $d_{n,k}$ we have made, these yield the so-called Padé rational approximations to e^z. That is, if $\bar{s}_n(z)$ is expanded in ascending powers of z, the resulting series will agree with the Taylor series for e^z to $2n + 1$ terms. Thus

$$s_2(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{144} + \cdots \tag{46}$$

which agrees with the Taylor series for e^z to five terms. It is not known in general how to choose the weights $\mu_{n,k}$ so that the Padé approximations to any given function will result. For a wide class of functions, namely, when $f(z)$ is a Gaussian hypergeometric function one of whose numerator parameters is one, and for limiting cases of this function, e^z being an example, this is known. Also for these functions, the numerator and denominator polynomials of the rational approximations can be conveniently calculated by means of three term recursion relations; for instance, for $Q_n(z)$ it can be shown

$$Q_{n+1}(z) = 2(2n + 1)Q_n(z) + z^2Q_{n-1}(z), \qquad n = 1, 2, \ldots \tag{47}$$

Let $z = 1$, to further specialize our findings; Eq. (44) then yields a sequence of approximations to e of astonishing accuracy.

$$\{s_k(1)\} = \{3, 2.714, 2.71831, 2.7182817, \ldots\}$$

the last entry being off in the seventh decimal place.

Richardson's Extrapolation to the Limit; Romberg Integration

In many practical situations, s_n is such that it converges to α (or diverges from α) in the following manner:

$$s_n = \alpha + r_n \tag{48}$$

$$r_n \sim n^\theta[c_1 + \frac{c_2}{n^\omega} + \frac{c_3}{n^{2\omega}} + \cdots], \qquad n \to \infty \qquad \omega > 0 \tag{49}$$

Let $n \to pn$ in Eq. (49). Then

$$s_{pn} = \alpha + r_{pn} \tag{50}$$

$$r_{pn} \sim p^\theta n^\theta \left[c_1 + \frac{c_2}{(pn)^\omega} + \frac{c_3}{(pn)^{2\omega}} + \cdots\right] \tag{51}$$

Multiplying Eq. (49) by p^θ and subtracting Eq. (51) gives

$$\frac{p^\theta s_n - s_{np}}{p^\theta - 1} = s_{n,1} = \alpha + r_{n,1} \tag{52}$$

$$r_{n,1} = n^{\theta-\omega}\left[c_{1,1} + \frac{c_{2,1}}{n^\omega} + \frac{c_{3,1}}{n^{2\omega}} + \cdots\right] \tag{53}$$

Letting $n \to np$ in Eq. (53) and repeating this we can define another sequence,

$$s_{n,2} = \frac{p^{\theta-\omega}s_{n,1} - s_{np,1}}{p^{\theta-\omega} - 1} = \alpha + r_{n,2} \tag{54}$$

and so on. Thus we have the double sequence $s_{n,m}$ defined by

$$s_{n,m+1} = \frac{p^{\theta-m\omega}s_{n,m} - s_{np,m}}{p^{\theta-m\omega} - 1} \tag{55}$$

It often happens that this sequence converges to α as $m \to \infty$ much faster than it does as $n \to \infty$. If, in addition, s_{np} may be calculated very easily once s_n is known, the algorithm may offer great advantages. Let $n \to p^n$ in Eq. (55) and define a new sequence $t_{n,m} = s_{p^n,m}$. Then the formula becomes

$$t_{n,m+1} = \frac{p^{\theta-m\omega}t_{n,m} - t_{n+1,m}}{p^{\theta-m\omega} - 1}, \qquad t_{n,0} = t_n = s_{p^n} \tag{56}$$

and this is usually the term the algorithm takes in practice. Obviously it can be applied to any sequence t_n. It is seen that

$$t_{n,m} = \sum_{k=0}^{m} \mu_{m,k} t_{n+k,0} \tag{57}$$

so Eq. (56) in reality is just a Toeplitz process applied to the sequence $t_{n,0}$ starting with $[t_n, t_{n+1}, \ldots]$ rather than $[t_0, t_1, \ldots]$. In fact if $\theta = -1$, $\omega = 1$, and $p = \sigma$, then it can be verified that the $\mu_{m,k}$ are those corresponding to the polynomial $P_m(\lambda)$ in line 4 of Table 2. It may also be shown that the procedure is regular in m (see Bauer *et al.* [7]) so

$$\lim_{m\to\infty} t_{n,m} = \alpha \qquad \text{if} \lim_{n\to\infty} t_n = \alpha \tag{58}$$

As an example, consider the problem of evaluating the integral

$$I = \int_0^1 f(x)\,dx \tag{59}$$

by the trapezoidal rule. Define

$$s_n = \frac{1}{n}\sum_{k=0}^{n}{}'' f\left(\frac{k}{n}\right) \tag{60}$$

where the double prime notation means the first and last terms in the sum are to be halved. Then

$$I = s_n + R_n \tag{61}$$

where, according to Krylov [8, ch. 11],

$$\left.\begin{aligned} R_n &\sim \frac{\alpha_1}{n^2} + \frac{\alpha_2}{n^4} + \frac{\alpha_3}{n^6} + \cdots \\ \alpha_1 &= -\frac{1}{12}[f'(1) - f'(0)] \\ \alpha_2 &= \frac{1}{720}[f'''(1) - f'''(0)], \text{etc.} \end{aligned}\right\} \tag{62}$$

provided that f has suitable differentiability properties. If we choose $\omega = 1$, $p = 2$, and $\theta = -2$ in Formula (56), this means we calculate $s_0, s_2, s_4, s_8, \ldots$, or, each time the number of intervals is doubled. But to calculate $s_{2^{n+1}}$ from s_{2^n} means computing the function f only at an additional 2^n points, since the other required 2^n values have already been computed. If we let

$$t_n = \frac{1}{2^n} \sum_{k=0}^{2^n}{}'' f\left(\frac{k}{2^n}\right), \qquad n = 0, 1, 2, \ldots \tag{63}$$

then we have

$$t_{n,m+1} = \frac{4^{m+1} t_{n+1,m} - t_{n,m}}{4^{m+1} - 1}, \qquad t_{n,0} = t_n \tag{64}$$

As an example, let

$$I = \int_0^1 \frac{dx}{(x+1)} = \ln 2 = 0.69314171805 \ldots \tag{65}$$

The results are shown in Table 4. For a more detailed discussion of this procedure

TABLE 4

n/j	0	1	2	3	4
0	0.750000000	0.694444444	0.693174603	0.693147479	0.693147181
1	0.708333333	0.693253967	0.693147901	0.693147182	
2	0.697023809	0.693154532	0.693147193		
3	0.694121851	0.693147652			
4	0.693391202				
5					

and generalizations of it, see the articles by Bauer *et al.* [7] and Bulirsch and Stoer [9, 10]. The general algorithm (64) is called Romberg quadrature.

Nonregular Methods

By assuming that s_n goes to zero in powers of $1/(n + \gamma)$, γ a real positive parameter, we can derive a class of nonregular Toeplitz methods. Let

$$s_n = \alpha + \frac{c_1}{(n+\gamma)} + \frac{c_2}{(n+\gamma)^2} + \cdots, \qquad \gamma > 0 \tag{66}$$

(This is, for all practical purposes, the Series 49 with $\theta = -1$, $\omega = 1$.) Letting $n = k$, multiplying both sides of the equation by $\mu_{n,k}$, and summing from $k = 0$ to n gives

$$\sum_{k=0}^{n} \mu_{n,k} s_n = \bar{s}_n = \alpha + \sum_{r=1}^{\infty} c_r \sum_{k=0}^{n} \frac{\mu_{n,k}}{(k+\gamma)^r} \tag{67}$$

provided, as usual, that we make

$$\sum_{k=0}^{n} \mu_{n,k} = 1 \tag{68}$$

Thus it would be logical to choose the $\mu_{n,k}$ so that the sum on the right of Eq. (67) vanishes to as many terms as possible. This means that

$$\sum_{k=0}^{n} \frac{\mu_{n,k}}{(k+\gamma)^r} = 0, \qquad r = 1, 2, \ldots, n \tag{69}$$

These n equations with Eq. (68) determine the $\mu_{n,k}$ uniquely. Solving these equations by Cramer's rule gives

$$\mu_{n,k} = (-1)^k \frac{\begin{vmatrix} \frac{1}{\gamma} & \frac{1}{\gamma+1} & \cdots & \frac{1}{\gamma+k-1} & \frac{1}{\gamma+k+1} & \cdots & \frac{1}{\gamma+n} \\ \frac{1}{\gamma^2} & \frac{1}{(\gamma+1)^2} & \cdots & \frac{1}{(\gamma+k-1)^2} & \frac{1}{(\gamma+k+1)^2} & \cdots & \frac{1}{(\gamma+n)^2} \\ \vdots & \vdots & & & & & \vdots \\ \frac{1}{\gamma^n} & \frac{1}{(\gamma+1)^n} & \cdots & \frac{1}{(\gamma+k-1)^n} & \frac{1}{(\gamma+k+1)^n} & \cdots & \frac{1}{(\gamma+n)^n} \end{vmatrix}}{\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \frac{1}{\gamma} & \frac{1}{\gamma+1} & \cdots & \frac{1}{\gamma+n} \\ \vdots & \vdots & & \vdots \\ \frac{1}{\gamma^n} & \frac{1}{(\gamma+1)^n} & \cdots & \frac{1}{(\gamma+1)^n} \end{vmatrix}} \tag{70}$$

Using the fact that

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_1^2 & \cdots & x_1^n \\ x_2 & x_2^2 & \cdots & x_2^n \\ \vdots & \vdots & & \vdots \\ x_n & x_n^2 & \cdots & x_n^n \end{vmatrix} = \prod_{j=1}^{n-1} \prod_{k=j+1}^{n} (x_k - x_j) \equiv V_n(x_1, x_2, \ldots, x_n) \tag{71}$$

(see Aitken [11]), we find

$$\mu_{n,k} = \frac{(-1)^{n+k}(\gamma + k)^n}{n\,!} \binom{n}{k} \tag{72}$$

It is easy to show that $T: \{s_n\} \to \{\bar{s}_n\}$ is not totally regular. Let s_n be given by the convergent sequence

$$s_n = (-1)^n \sigma^n, \qquad 0 < \sigma < 1 \tag{73}$$

Then if we use Stirling's formula we find

$$\begin{aligned} |\bar{s}_n| &= \frac{1}{n\,!} \sum_{k=0}^{n} \sigma^k (\gamma + k)^n \binom{n}{k} > \frac{\sigma^n(\gamma + n)^n}{n\,!} \\ &> \frac{M_{n_0}}{n^{\frac{1}{2}}} (\sigma e)^n, \qquad n > n_0 \end{aligned} \tag{74}$$

and if $\sigma > 1/e$, the right-hand side of Eq. (74) is a divergent sequence. However, T is regular and an acceleration process for all sequences of the form of Eq. (66) if

$$M(\gamma) = \sum_{r=1}^{\infty} \frac{|c_r|}{\gamma^r} < \infty \tag{75}$$

and for these cases we can compute a bound for $\bar{r}_n$.

We have

$$\bar{r}_n = \sum_{r=1}^{\infty} c_{r+n} \sum_{k=0}^{n} \frac{\mu_{n,k}}{(k + \gamma)^{r+n}} \tag{76}$$

Now we use

$$(k + \gamma)^{-r} = \frac{1}{\Gamma(r)} \int_0^\infty e^{-(k+\gamma)t} t^{r-1}\, dt, \qquad r = 1, 2, \ldots \tag{77}$$

and we find that

$$\begin{aligned} |\bar{r}_n| &= \frac{1}{n\,!} \left| \sum_{r=1}^{\infty} \frac{c_{r+n}}{\Gamma(r)} \int_0^\infty t^{r-1} \sum_{k=0}^{n} \binom{n}{k} (-1)^k e^{-(\gamma+k)t}\, dt \right| \\ &= \frac{1}{n\,!} \left| \sum_{r=1}^{\infty} \frac{c_{r+n}}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-\gamma t} (1 - e^{-t})^n\, dt \right| \qquad (78) \\ &\le \frac{1}{n\,!} \sum_{r=1}^{\infty} \frac{|c_{r+n}|}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-\gamma t}\, dt \\ &= \frac{1}{n\,!} \sum_{r=1}^{\infty} \frac{|c_{r+n}|}{\gamma^r} \le \frac{\lambda^n}{n!} M(\gamma) \qquad (79) \end{aligned}$$

Thus we can find a constant $c(\gamma)$ and an integer k such that

$$\frac{|\bar{r}_n|}{|r_n|} \le \frac{\gamma^n c(\gamma) n^k}{n\,!} \tag{80}$$

in fact, k corresponds to the first $c_k \neq 0$. Thus sequences of the form of Eq. (66) which converge algebraically in n are transformed into sequences which converge roughly like $1/n!$, a tremendous computational gain. Examples of the application of $\mu_{n,k}$ for $\gamma = 1$ to test sequences t_n are given in Tables 5, 6, and 7. Despite the fact that

TABLE 5

$$s_n = \sum_{k=0}^{n} (-1)^k (k+1), \text{ divergent}$$

$$\bar{s}_n = \sum_{k=0}^{n} \mu_{n,k} s_k$$

n	Cesaro	Binomial	Chebyshev	Salzer
0	1	1	1	1
1	0	0	−0.3333333333	−3
2	$\frac{2}{3}$	0.25	+0.5294117647	13.5
3	0	0.25	0.1515151515	−52.5
4	$\frac{3}{5}$	0.25	0.2790294627	206.6666666
5	0	0.25	0.2423431460	−797.5333333
6	$\frac{4}{7}$	0.25	0.2518749043	3055.033333
7	0	0.25	0.2495645247	—
8	$\frac{5}{9}$	0.25	0.2500972431	—
9	0	0.25	0.2499789353	—
10	$\frac{6}{11}$	0.25	0.2500044541	—

TABLE 6

$$s_n = \sum_{k=0}^{n} \frac{(-1)^k}{k+1}, \qquad \alpha = \ln 2 = 0.69314718$$

$$\bar{s}_n = \sum_{k=0}^{n} \mu_{n,k} s_k$$

n	Cesaro	Binomial	Chebyshev	Salzer
0	1	1	1	1
1	0.75	0.75	0.66666667	0
2	0.77777778	0.70833333	0.68627451	2.25
3	0.72916667	0.69791667	0.69360269	−3.1944444
4	0.74	0.69479167	0.69312536	11.093750
5	0.71944444	0.69375000	0.69315096	−28.475417
6	0.72517007	0.69337798	0.69314685	—
7	0.71383929	0.69323847	0.69314723	—
8	0.71737213	0.69318421	0.69314717	—
9	0.71019841	0.69316251	0.69314718	—
10	0.71259347	0.69315363	0.69314718	—

TABLE 7

$$s_n = \sum_{k=0}^{n} \frac{1}{(k+1)^2}, \qquad \alpha = \frac{\pi^2}{6} = 1.6449341\ldots$$

$$\bar{s}_n = \sum_{k=0}^{n} \mu_{n,k}\, s_k$$

n	Cesaro	Binomial	Chebyshev	Salzer
0	1	1	1	1
1	1.125	1.125	1.1666667	1.5
2	1.2037037	1.2152778	1.2875817	1.625
3	1.2586806	1.2821181	1.3574635	1.6435185
4	1.2996667	1.3327951	1.4047911	1.6449653
5	1.3316204	1.3720833	1.4390134	1.6449514
6	1.3573599	1.4031748	1.4648380	1.6449352
7	1.3786177	1.4282444	1.4849778	1.6449339
8	1.3965232	1.4488021	1.5011046	1.6449341
9	1.4118477	1.4659163	1.5142992	1.6449341
10	1.4251371	1.4803564	1.5252896	1.6449341

$[\mu_{n,k}]$ is not regular, Table 7 shows it can be a very powerful method, and in this case outperforms the regular methods.

For further details on generalizations, the reader should consult Salzer [12, 13], Salzer and Kimbro [14], and Wimp [15].

SCALAR ALGORITHM: NONLINEAR CASE

Again the sequence under consideration will be one of complex numbers $\{s_n\}$, but the transformation T will not have the linearity property 5. These transformations constitute one of the most effective classes of algorithms in all of numerical analysis.

The Schmidt Transformation

The heuristic derivation of this class of transformations is as follows. In many sequences that arise in applied mathematics, the sequence converges to its limit α as though it were composed of its limit and a linear combination of exponential "transients," i.e., terms that decay exponentially in n. These are precisely the Sequences (21) considered previously. Let

$$s_n = \alpha + \sum_{r=1}^{m} c_r \lambda_r{}^n, \quad c_j \neq 0;\ \lambda_j \neq 0, 1;\quad \lambda_i \neq \lambda_j,\quad i \neq j \tag{81}$$

Then

$$r_n = \sum_{r=1}^{m} c_r \lambda_r^{\,n} \tag{82}$$

It is shown in books on difference equations (see, for instance, Milne-Thomson [16]) that there is a unique linear difference operator of order m that annihilates r_n, i.e., there exist unique constants $A_1, A_2, \ldots, A_m$ such that

$$r_n + A_1 r_{n+1} + A_2 r_{n+2} + \cdots + A_m r_{n+m} = 0 \tag{83}$$

for all n. Replacing n by $n + j$, $1 \leq j \leq m$, gives, with the above equation, $m + 1$ equations in m unknowns, $A_1, A_2, \ldots, A_m$. Such a system will possess a solution if and only if the matrix composed of the coefficients augmented by the constant column vector has rank $< m + 1$, i.e., if and only if

$$\begin{vmatrix} r_n & \cdots & r_{n+m} \\ r_{n+1} & \cdots & r_{n+m+1} \\ \vdots & & \vdots \\ r_{n+m} & \cdots & r_{n+2m} \end{vmatrix} = 0 \tag{84}$$

Letting $r_n = s_n - \alpha$ and decomposing the above determinants by column manipulations (see Aitken [11]) gives

$$\alpha = W_m(s_n)/U_m(s_n) \tag{85}$$

$$W_m(s_n) = \begin{vmatrix} s_n & \Delta s_n & \cdots & \Delta s_{n+m-1} \\ s_{n+1} & \Delta s_{n+1} & \cdots & \Delta s_{n+m} \\ \vdots & \vdots & & \vdots \\ s_{n+m} & \Delta s_{n+m} & \cdots & \Delta s_{n+2m-1} \end{vmatrix} \tag{86}$$

$$U_m(s_n) = \begin{vmatrix} 1 & \Delta s_n & \cdots & \Delta s_{n+m-1} \\ 1 & \Delta s_{n+1} & \cdots & \Delta s_{n+m} \\ \vdots & \vdots & & \vdots \\ 1 & \Delta s_{n+m} & \cdots & \Delta s_{n+2m-1} \end{vmatrix} \tag{87}$$

Notice that the A_j in Eq. (83) are simply the symmetric functions of $\lambda_1, \ldots, \lambda_m$, i.e.,

$$A_1 = -(\lambda_1 + \lambda_2 + \cdots + \lambda_m), \qquad A_2 = (\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + \cdots + \lambda_{m-1}\lambda_m)$$
$$A_m = (-1)^m \lambda_1 \lambda_2 \cdots \lambda_m \tag{88}$$

Now, in general, s_n will not be of the form of Eq. (81). However, we may use Eq. (85) to *define* a new sequence $s_{n,m}$,

$$T_m(s_n) = s_{n,m} = \frac{W_m(s_n)}{U_m(s_n)}, \qquad m = 0, 1, 2, \ldots \tag{89}$$

and we get a class of transformations, one for each value of m. We would expect that, in general, the larger m is, the more accelerative T_m is, and we would also expect T_m to sum exactly sequences of the form of Eq. (81). The latter statement is true, as we shall later see, but the first needs much qualification because T_m is not totally regular (see Shanks [17] for examples). T_m is not linear, but it has the quasi-linearity properties

$$T_m(cs_n) = cT_m(s_n) \tag{90}$$

$$T_m(c + s_n) = c + T_m(s_n) \tag{91}$$

For a development of the theory of $T_m(s_n)$, see Shanks [17] and Schmidt [18].

For $m = 1$, $T_1(s_n)$ yields the well-known Aitken's δ^2 process.

$$s_{n,1} = \frac{s_n s_{n+2} - s_{n+1}^2}{\Delta^2 s_n} \tag{92}$$

and for $m = 2$ we have

$$s_{n,2} = \frac{s_n s_{n+2} s_{n+4} - s_n s_{n+3}^2 - s_{n+1}^2 s_{n+4} + 2 s_{n+1} s_{n+2} s_{n+3} - s_{n+2}^3}{\Delta^2 s_n \Delta^4 s_n - (\Delta^3 s_n)^2} \tag{93}$$

Of course, the larger m is the more difficult $s_{n,m}$ is to compute. (Wynn, as we shall presently see, has devised a formalism, though, which makes the computation of $T_m(s_n)$ very easy.) Another alternative is to define a new transformation as an iteration of T_1, i.e.,

$$\begin{aligned} T_m^*(s_n) = s_{n,m}^* &= \underbrace{T_1(T_1(\cdots(T_1(s_n))\cdots))}_{m \text{ times}} \\ &= T_1{}^m(s_n) \end{aligned} \tag{94}$$

Thus $T_m^*(s_n)$ is the result of applying Aitken's δ^2 process repeatedly to the sequence s_n, with $s_{n,0}^* = s_n$.

An example, given by Shanks [17], of the application of T_m^* to the sequence is shown in Table 8.

Note that both T_m and T_m^*, when applied to s_n, yield a *double array*, where m and n both take on positive integral values. For regularity we require that $T_m(s_n)$ or $T_m^*(s_n)$ approach α as $n \to \infty$ if s_n converges to α. But there is another type of convergence possible, namely, horizontal convergence. In other words, if $s_n \to \alpha$, does

$$\lim_{m\to\infty} s_{n,m} = \alpha, \qquad \lim_{m\to\infty} s_{n,m}^* = \alpha? \tag{95}$$

In general, horizontal convergence is the more rapid. For linear transformations of the Toeplitz type, as we have seen, simple conditions on the $\mu_{n,k}$ assure the total regularity of T. If $s_n \to \alpha$, $\{s_n\} \in S$, no simple conditions on S are known which guarantee that $s_{n,m}$ or $s_{n,m}^* \to \alpha$ either as $n \to \infty$ or as $m \to \infty$. For certain sequence spaces (Eq. 81 or Eq. 105 below), some information about the behavior of T_n and T_n^* is known. But it is not even known whether T_m is totally regular (as $n \to \infty$). (It is known that T_n^* is not regular (as $n \to \infty$). See Shanks [17].) Questions about horizontal convergence of T_n can sometimes be decided by an appeal to the

TABLE 8

$s_n = 2v_n(1, \frac{1}{2}, -1) = 4 \sum_{k=0}^{n} \frac{(-1)^k}{(2k+1)}$

$\alpha = \lim_{n\to\alpha} s_n = \pi = 3.1415926535\ldots$

	$s^*_{n,m}$				
n/m	0	1	2	3	4
1	4.0000000				
2	2.6666667	3.1666667			
3	3.4666667	3.1333333	3.1421053		
4	2.8959381	3.1452381	3.1414502	3.1415993	
5	3.3396825	3.1396825	3.1416433	3.1415909	3.1415928
6	2.9760462	3.1427129	3.1415713	3.1415933	3.1415927
7	3.2837385	3.1408814	3.1416029	3.1415925	
8	3.0170718	3.1420718	3.1415873		
9	3.2523659	3.1412548			
10	3.0418396				

theory of continued fractions, see the section entitled Rational Approximations. We now wish to examine the effect of T_m on an important class of sequences. Let

$$\begin{gathered} s_n = \alpha + r_n \\ r_n = c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_m\lambda_m^n \\ \lambda_j \neq 0, 1, \qquad c_j \neq 0, \qquad \lambda_i \neq \lambda_j, \qquad i \neq j \end{gathered} \tag{96}$$

Consider $W_m(r_n)$. Since there exist constants A_j such that Eq. (81) holds, we can multiply the second row of W_m by A_1, the third by $A_2, \ldots$, where the A_j are as in Eq. (88), and add all these rows to the first, which produces a determinant whose first row is zero and hence is zero. Performing the same operation on $U_m(r_n)$ yields

$$U_m(r_n) = (1 + A_1 + A_2 + \cdots + A_m) \begin{vmatrix} \Delta r_{n+1} & \cdots & \Delta r_{n+m} \\ \vdots & & \vdots \\ \Delta r_{n+m} & \cdots & \Delta r_{n+2m-1} \end{vmatrix} \tag{97}$$

The first factor cannot be zero since, by Eq. (88),

$$\begin{aligned} 1 + A_1 + \cdots + A_m &= (\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_m)\,|_{\lambda=1} \\ &\neq 0 \end{aligned} \tag{98}$$

by the hypothesis $\lambda_j \neq 1$. Also the determinant above can be factored as

$$[V_m(\lambda_1, \lambda_2, \ldots, \lambda_m)]^2 \prod_{j=1}^{m} c_j(\lambda_j - 1)\lambda_j^{n+1} \neq 0 \tag{99}$$

by our hypotheses. V_m is as defined in Eq. (71).

Thus T_m is regular and accelerative, in fact exact, for the Sequences (81).

By elementary row and column manipulations, we find

$$W_m(s_n) = \begin{vmatrix} s_n & \Delta s_n & \cdots & \Delta^m s_n \\ \Delta s_n & \Delta^2 s_n & \cdots & \Delta^{m+1} s_n \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \Delta^m s_n & \Delta^{m+1} s_n & \cdots & \Delta^{2m} s_n \end{vmatrix} \tag{100}$$

$$U_m(s_n) = W_{m-1}(\Delta^2 s_n) \tag{101}$$

so

$$s_{n,m} = W_m(s_n)/W_{m-1}(\Delta^2 s_n) \tag{102}$$

Using the facts that

$$W_m(\lambda^n s_n) = \lambda^{(m+1)(n+m)} W_m(s_n) \tag{103}$$

and that

$$\Delta^j n^\theta \left\{ c_1 + \frac{c_2}{n} + \cdots \right\} = \theta(\theta - 1) \cdots (\theta - j + 1) n^{\theta - j} \left[c_1 + 0\left(\frac{1}{n}\right) \right], \quad n \to \infty \tag{104}$$

where the first series is an asymptotic series in n, we find that for sequences of the form

$$s_n \sim \alpha + \lambda^n n^\theta \left[c_1 + \frac{c_2}{n} + \cdots \right] \tag{105}$$

$T_m(s_n)$ can be estimated by

$$T_m(s_n) = s_{n,m} = \alpha + \frac{\lambda^{2m+n} n^{\theta-2m} c_1 m!\,(-\theta)_m}{(\lambda - 1)^{2m}} \left[1 + 0\left(\frac{1}{n}\right) \right]$$

$$n \to \infty, \qquad \lambda \neq 0, 1; \qquad \theta \neq 0, 1, 2, \ldots, m - 1, \qquad c_1 \neq 0 \tag{106}$$

As an example of Eq. (106), consider the sequence

$$s_n = \sum_{k=0}^{n} \frac{(-1)^k}{(k+1)(k+2)}, \qquad s_n \to 2 \ln 2 - 1 = \alpha \tag{107}$$

It can be shown that

$$s_n = \alpha + \frac{(-1)^n}{n^2} \left[\frac{1}{2} + \frac{c_2}{n} + \cdots \right], \qquad n \to \infty \tag{108}$$

Thus

$$s_{n,m} = \alpha + \frac{(-1)^n\, m!\,(m+1)!}{2^{2m+1} n^{2m+2}} \left[1 + 0\left(\frac{1}{n}\right) \right], \qquad n \to \infty \tag{109}$$

and for $m > 0$ the improvement in convergence of the series is obvious.

Wynn [19] has shown that if $\lambda = 1$, then

$$T_m(s_n) = \alpha + \frac{M}{n} [1 + 0(n^{-1})], \qquad n \to \infty \tag{110}$$

while if s_n is given by an asymptotic sequence more general than Eq. (81), namely,

$$s_n \sim \alpha + \sum_{r=1}^{\infty} c_r \lambda_r^n, \qquad 1 > |\lambda_1| > |\lambda_2| > \cdots > 0 \tag{111}$$

then

$$T_m(s_n) \sim \alpha + c_{m+1}\lambda_{m+1}^n \prod_{j=1}^{m} \left| \frac{\lambda_{m+1} - \lambda_j}{1 - \lambda_j} \right|^2 \tag{112}$$

which shows that the effect of T_m is to remove from s_n the first m largest transients $\lambda_1, \ldots, \lambda_m$.

If s_n are the partial sums of the Taylor series for a function $f(z)$ analytic at $z = 0$,

$$s_n = \sum_{k=0}^{n} a_k z^k \tag{113}$$

then $T_m(s_n)$ gives a rational approximation to $f(z)$. Shanks [17] has proved *if $f(z)$ is a ratio of two polynomials with no common factors whose numerator is of degree q and denominator of degree p, then if $m > p$, $n + m > q$, $T_m(s_n) = f(z)$.*

The Epsilon Algorithm

The epsilon algorithm, discovered by Wynn [20], is actually a systematic and very efficient way of computing the Schmidt transformation (85). Let

$$\begin{aligned} \varepsilon_{2m}^{(n)} &= T_m(s_n) \\ \varepsilon_{2m+1}^{(n)} &= 1/T_m(\Delta s_n), \qquad m, n = 0, 1, 2, \ldots \end{aligned} \tag{114}$$

Then Wynn showed that $\varepsilon_m^{(n)}$ satisfies

$$\begin{aligned} \varepsilon_{m+1}^{(n)} &= \varepsilon_{n-1}^{(n+1)} + \frac{1}{\varepsilon_m^{(n+1)} - \varepsilon_m^{(n)}}, \qquad m, n = 0, 1, 2, \ldots \\ \varepsilon_{-1}^{(n)} &= 0 \end{aligned} \tag{115}$$

where $\varepsilon_0^{(n)} = s_n$. Thus given s_n one can systematically fill out the array

$$\begin{array}{ccccc}
0 & & & & \\
 & \varepsilon_0^{(0)} & & & \\
0 & & \varepsilon_1^{(0)} & & \\
 & \varepsilon_0^{(1)} & & \varepsilon_2^{(0)} & \\
0 & & \varepsilon_1^{(1)} & & \varepsilon_3^{(0)} \\
 & \varepsilon_0^{(2)} & & \varepsilon_2^{(1)} & \\
0 & & \varepsilon_1^{(2)} & & \varepsilon_3^{(1)} \\
 & \varepsilon_0^{(3)} & & \varepsilon_2^{(2)} & \\
0 & & \varepsilon_1^{(3)} & & \varepsilon_3^{(2)} \\
\cdot & \varepsilon_0^{(4)} & & \varepsilon_2^{(3)} & \cdot \\
\cdot & \cdot & \varepsilon_1^{(4)} & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot
\end{array}$$

and any $\varepsilon_j^{(k)}$ is connected by Formula (115) with three other epsilons lying in a lozenge or rhombus as indicated by the black lines in the array. Thus horizontal convergence of the $T_m(s_n)$ algorithm is equivalent to requiring that the diagonal $\varepsilon_k^{(n)}$ with even subscript approach a limit as $k \to \infty$,

$$\lim_{k \to \infty} \varepsilon_{2k}^{(n)} = \alpha, \qquad \text{if} \lim_{n \to \infty} s_n = \alpha \tag{116}$$

There is an intimate connection between the epsilon algorithm and the continued fraction development for certain functions defined by Stieltjes integrals (see Widder [21] for a definition of this integral). Since this connection serves to delineate the classes of sequences for which $T_m(s_n)$ is regular in the sense that

$$\lim_{m \to \infty} T_m(s_n) = \alpha \qquad \text{if} \lim_{n \to \infty} s_n = \alpha \tag{117}$$

we will explore this further. This diagonal convergence of the epsilon algorithm is generally the more rapid convergence, since it is the result of applying successively higher order transformations to s_n.

Suppose the s_n are the partial sums of a formal power series in $1/z$ of the form

$$s_n(z) = \sum_{k=0}^{n-1} \frac{a_k}{z^{k+1}}, \qquad n = 1, 2, 3, \ldots \tag{118}$$

$$s_0(z) = 0$$

Then it can be shown that $\varepsilon_{2k}^{(0)}$ is the result of truncating a formal continued fraction

$$\frac{a_0|}{|z} - \frac{q_1|}{|1} - \frac{e_1|}{|z} - \frac{q_2|}{|1} - \frac{e_2|}{|z} - \cdots \tag{119}$$

to yield a rational function

$$\varepsilon_{2k}^{(0)} = \frac{a_0|}{|z} - \frac{q_1|}{|1} - \frac{e_1|}{|z} - \cdots - q_k \tag{120}$$

Thus $\lim_{k \to \infty} \varepsilon_{2k}^{(0)}$ will exist when the continued fraction (119) converges. This will be true if there exists a unique bounded nondecreasing function $\psi(t)$ constant for $t < 0$ such that

$$a_i = \int_{-\infty}^{\infty} t^i \, d\psi(t) \tag{121}$$

the latter being a Stieltjes integral. Then the continued fraction (119) will converge to

$$F(z) = \int_{-\infty}^{\infty} \frac{d\psi(t)}{z - t} \tag{122}$$

for all z except on the positive real axis. There are conditions on the a_i of a complicated nature involving Hankel determinants that will guarantee the convergence of (119), see Wall [22, ch. 14].

As an example, let

$$\psi(t) = e^{-t}, \qquad t > 0 \tag{123}$$

$$a_j = j\,! = \int_0^{\infty} t^j e^{-t} \, dt \tag{124}$$

Then $\varepsilon_{2k}^{(0)}(z)$ converges to

$$F(z) = -\int_0^\infty \frac{e^{-t}}{z-t}\,dt \tag{125}$$

if z is not on the positive real axis. In the example in Table 9 we let $z = -1$, so

$$\alpha = \int_0^\infty \frac{e^{-t}\,dt}{1+t} = 0.5963473\ldots \tag{126}$$

It is interesting to note that for all values of z, the sequence $s_n(z)$

$$s_n(z) = \sum_{k=0}^{n-1} \frac{k!}{z^{k+1}}, \qquad n = 1, 2, \ldots \tag{127}$$
$$s_0(z) = 0$$

is divergent. We know of no linear methods which will sum so wildly divergent a sequence. For this example, the sequence $\varepsilon_{2k}^{(0)}$ is

$$0,\ 0.5,\ 0.571429,\ 0.588235,\ 0.593301,\ \ldots$$

$\varepsilon_8^{(0)}$ having an error of about 8.1×10^{-3}. For the sequence

$$s_n = \sum_{k=0}^{n} \frac{(-1)^k}{k+1} \to \ln 2 = 0.693147\ldots \tag{128}$$

the sequence $\varepsilon_{2k}^{(0)}$ is

$$1,\ 0.7,\ 0.693269,\ 0.6931525,\ \ldots$$

the last number having an error of 5.4×10^{-6}. On sequences which approach their limits monotonically, however, the epsilon algorithm is usually much less satisfactory. If

$$s_n = \sum_{k=1}^{n+1} \frac{1}{k^2} \to \frac{\pi^2}{6} = 1.6449\ldots \tag{129}$$

then $\varepsilon_{2k}^{(0)}$ is 1, 1.45, 1.549, 1.588, ... The method of the section entitled Nonregular Methods is better adapted to such sequences.

It must be remembered that in contrast to the linear Toeplitz methods, the nonlinear transformations so far considered are not totally regular, and it is almost impossible to analyze the error involved. Thus caution is required in their use.

The epsilon algorithm is related to the Rutishauser quotient–difference algorithm, which has been the subject of a vast literature, see Rutishauser [23] and Henrici [24]. For further material on nonlinear sequence transformations based on continued fraction theory, see Bauer [25].

Other Nonlinear Transformations

Let P be a difference operator, linear or nonlinear, which annihilates constants and has the property that if $s_n \to \alpha$, then

$$\lim_{n\to\infty} P(s_n) = P(\alpha) = 0 \tag{130}$$

TABLE 9

$$S_n = \sum_{k=0}^{n-1} k!\,(-1)^k, \qquad s_0 = 0$$

s_n	$\varepsilon_1^{(n)}$	$\varepsilon_2^{(n)}$	$\varepsilon_3^{(n)}$	$\varepsilon_4^{(n)}$	$\varepsilon_5^{(n)}$	$\varepsilon_6^{(n)}$	$\varepsilon_7^{(n)}$	$\varepsilon_8^{(n)}$
0								
	1							
1		$\frac{1}{2}$						
	-1		5					
0		$\frac{2}{3}$		$\frac{4}{7}$				
	$\frac{1}{2}$		$-\frac{11}{2}$		$\frac{69}{4}$			
2		$\frac{1}{2}$		$\frac{8}{13}$		$\frac{10}{17}$		
	$-\frac{1}{6}$		$\frac{19}{6}$		$-\frac{235}{12}$		$\frac{1777}{36}$	
-4		$\frac{4}{5}$		$\frac{4}{7}$		$\frac{44}{73}$		0.593301 . . .
	$\frac{1}{24}$		$-\frac{29}{24}$		$\frac{593}{48}$		$-\frac{8149}{144}$	
20		0		$\frac{20}{31}$		$\frac{10}{17}$		
	$-\frac{1}{120}$		$\frac{41}{120}$		$-\frac{417}{80}$			
-100		$\frac{20}{7}$		$\frac{20}{43}$				
	$\frac{1}{720}$		$-\frac{11}{144}$					
620		-10						
	$-\frac{1}{5040}$							
-4420								

Let $f(x)$ be a function with the property that

$$f(P(s_{n+j})) = s_{n+j}, \qquad j = 0, 1, 2, \ldots \tag{131}$$

By Lagrange's interpolation formula

$$f(x) = \sum_{k=0}^{m} s_{n+k} \prod_{j=0\neq k}^{m} \left[\frac{x - P(s_{n+j})}{P(s_{n+k}) - P(s_{n+j})}\right] + R_m(x, f) \tag{132}$$

Now by Eqs. (130) and (131)

$$f(0) = f(P(\alpha)) = \alpha \tag{133}$$

so if we put $x = 0$ in Eq. (132) and let

$$s_{n,m} = \alpha - R_m(0, f) \tag{134}$$

we have the class of sequence-to-sequence transformations

$$T_m\colon s_n \to s_{n,m}$$
$$s_{n,m} = \sum_{k=0}^{n} s_{n+k} \prod_{j=0\neq k}^{m} \left[1 - \frac{P(s_{n+k})}{P(s_{n+j})}\right]^{-1} \tag{135}$$

The simple case $P = \Delta$ yields

$$s_{n,m} = \sum_{k=0}^{m} s_{n+k} \prod_{j=0\neq k}^{m} \left[1 - \frac{\Delta s_{n+k}}{\Delta s_{n+j}}\right] \tag{136}$$

When the above transformation is used to define certain iterative methods for solving equations, it is known that the order of convergence is that of the Schmidt transformation, see Wimp [26]. However, its general convergence properties have not yet been studied.

CONTINUOUS TRANSFORMATIONS AND SPECIAL SEQUENCES, OPERATOR EQUATIONS

We have seen how assuming certain properties of the scalar sequence s_n leads naturally to classes of acceleration methods $T_m(s_n) = s_{n,m}$, where m takes on integer values. In many problems it is useful to devise acceleration methods which depend *continuously* on a parameter δ

$$T_\delta(s_n) = s_{n,\delta} \tag{137}$$

Then an optimal value of δ is chosen so that

$$\lim_{n\to\infty} \left|\frac{s_{n,\delta} - \alpha}{s_n - \alpha}\right| = 0 \tag{138}$$

as rapidly as possible. This method is particularly useful when one is working with sequences in an abstract vector space or algebra, where the linear or nonlinear transformations previously discussed for scalar equations either involve operations that are not defined (such as reciprocating a member of the sequence) or involve a prohibitive amount of computational effort (see the comments at the beginning of the section entitled Scalar Algorithm: Linear Case). Although such methods are usually limited in the improvement in convergence that can be attained (and for that reason are seldom applied to scalar sequences when the previous methods can be used), nevertheless they have the attraction of computational simplicity. Usually the sequence s_n is one which is constructed to converge to a desired root of an equation. It may converge too slowly or not at all, and then one hopes that a judicious choice of δ will improve convergence. The fact that these methods are applied to special sequences specifically designed to converge to a given quantity makes these methods conceptually different from the methods previously considered, which were applicable to large classes of sequences. Also, it is usually not possible, or even desirable, to write $T_\delta(s_n)$ in terms of s_n but rather, a rule is given for its formation. These methods are only implicitly sequence-to-sequence transformations.

As an example, suppose we wish to solve the scalar equation

$$x = g(x) \tag{139}$$

The method of functional iteration defines the sequence s_n by s_0 arbitrary and

$$s_{n+1} = g(s_n), \qquad n = 0, 1, 2, \ldots \tag{140}$$

It is then hoped that s_n converges to the desired root of Eq. (139). Let that root be α, so

$$\alpha = g(\alpha) \tag{141}$$

Usually s_n, if it converges, converges very slowly. Now, suppose we define a sequence $s_{n,\delta}$ by $s_{0,\delta} = s_0$ and

$$s_{n+1,\delta} = \delta g(s_{n,\delta}) + (1 - \delta)s_{n,\delta}, \qquad n = 0, 1, 2, \ldots \tag{142}$$

Denote the ρ neighborhood of α by

$$N(\rho, \alpha) = \{x \mid |x - \alpha| < \rho\} \tag{143}$$

Assume ρ is such that Eq. (139) has only one root, α, in $N(\rho, \alpha)$. By Taylor's theorem, if g is *continuously* differentiable in $N(\rho, \alpha)$,

$$g(s_{n,\delta}) = \alpha + (s_{n,\delta} - \alpha)g'(\xi_n) \tag{144}$$

where ξ_n is between α and $s_{n,\delta}$. Let

$$r_{n,\delta} = s_{n,\delta} - \alpha \tag{145}$$

Then

$$r_{n+1,\delta} = r_{n,\delta}(1 - \delta) + \delta g'(\xi_n) \tag{146}$$

Let δ be chosen such that

$$-2 < \delta(g'(x) - 1) < 0 \tag{147}$$

for all $x \in N(\rho, \alpha)$. This is always possible if ρ is sufficiently small and $g'(\alpha) \neq 1$. Then it is easily established by induction that if $s_0 \in N(\rho, \alpha)$, so is $s_{n,\delta}, n = 1, 2, \ldots$. Furthermore $s_{n,\delta}$ converges to α since

$$|r_{n+1,\delta}| \leq |r_{n,\delta}|\, d \tag{148}$$

implies

$$|r_{n,\delta}| \leq |s_0 - \alpha|\, d^n \tag{149}$$

where

$$d = \sup_{x \in N(\rho,\alpha)} |1 - \delta + \delta g'(x)| < 1 \tag{150}$$

The optimum value of δ is that which makes d smallest. If

$$g'(\alpha) = K \neq 1 \tag{151}$$

then this optimum value is, approximately,

$$\delta = \frac{1}{1 - K} \tag{152}$$

and with this choice of δ and s_0 sufficiently close to α, $s_{n,\delta}$ will converge exponentially in n. Exactly the same kind of analysis applies to the solution of vector Eq. (139) where, say, x is an n-dimensional vector and g is a vector function. For details, the reader should consult Isaacson and Keller [27]. The method may also be adopted to solve operator equations of a very general type. Let $B(X, X)$ be the space of all bounded linear operators which map a complete normed linear space X into itself. The norm of the operator $B \in B(X, X)$ is

$$\| B \| = \sup_{\|x\|=1} \| Bx \| \tag{153}$$

For any vector y,

$$\| By \| \leq \| B \| \; \| y \| \tag{154}$$

We say B is regular if there exists a $C \in B(X, X)$ such that

$$BC = CB = I \tag{155}$$

where I is the identity operator, $Ix = x$, for all $x \in X$. We define the spectrum of B to be the set of all (complex) λ such that $\lambda I - B$ is not regular. Suppose we wish to solve the equation

$$Ax = f, \qquad x, f \in X, \qquad A \in B(X, X) \tag{156}$$

for x, given f and A, and that A is regular.

We construct a sequence $s_{n,\delta}$ of members of X which we hope converges to the solution, x, of Eq. (156). Let $r_{n,\delta} = s_{n,\delta} - x$. First we define a splitting of the operator A by

$$A = N_0 - P_0 \tag{157}$$

where N_0 is regular. (This is usually done in such a way that N_0^{-1} is easily computed.) Next, let

$$N = (\delta + 1)N_0, \qquad P = N - A = P_0 + \delta N_0 \tag{158}$$

where δ is a real parameter to be chosen, and define

$$Ns_{n+1,\delta} = Ps_{n,\delta} + f, \qquad n = 0, 1, 2, \ldots \tag{159}$$

with $s_{0,\delta} = s_0$ arbitrary. Subtracting Eq. (156) from Eq. (159) gives

$$Nr_{n+1,\delta} = Pr_{n,\delta}, \qquad n = 0, 1, 2, \ldots \tag{160}$$

or

$$r_{n+1,\delta} = \left[\frac{N_0^{-1}P_0 + \delta I}{\delta + 1}\right] r_{n,\delta}, \qquad n = 0, 1, 2, \ldots \tag{161}$$

By induction we may show

$$r_{n,\delta} = \left[\frac{N_0^{-1}P_0 + \delta I}{\delta + 1}\right]^n r_{0,\delta} \tag{162}$$

and so

$$\| r_{n,\delta} \| < \left\| \frac{N_0^{-1}P_0 + \delta I}{\delta + 1} \right\|^n \| r_{0,\delta} \| \tag{163}$$

The spectral radius of an operator B is defined to be

$$\nu(B) = \lim_{n\to\infty} \| B^n \|^{1/n} \tag{164}$$

It may be shown that the limit exists and

$$\nu(B) = \sup_{\lambda \in Sp(B)} |\lambda| \tag{165}$$

see Taylor [28]. Therefore, given $\varepsilon > 0$ there exists an n_0 such that

$$\| r_{n,\delta} \| \leq \left| \nu\left(\frac{N_0^{-1}P_0 + \delta I}{\delta + 1}\right) + \varepsilon \right|^n \| r_{0,\delta} \|, \qquad n > n_0 \tag{166}$$

Thus $r_{n,\delta}$ will converge to x for all δ for which

$$\nu\left(\frac{N_0^{-1}P_0 + \delta I}{\delta + 1}\right) < 1 \tag{167}$$

Now

$$Sp\left(\frac{N_0^{-1}P_0 + \delta I}{\delta + 1}\right) = \left\{\mu \,\middle|\, \frac{\lambda + \delta}{\delta + 1} = \mu,\ \lambda \in Sp(N_0^{-1}P_0)\right\} \tag{168}$$

so the Requirement (167) in view of Eq. (168) becomes

$$\sup_{\lambda \in Sp(N_0^{-1}P_0)} \left|\frac{\lambda + \delta}{\delta + 1}\right| < 1 \tag{169}$$

As an example, suppose λ is real and $\lambda < 1$. Then this requirement for convergence is satisfied if

$$\delta > -\frac{\lambda + 1}{2} > -1 \qquad \text{for all } \lambda \in Sp(N_0^{-1}P_0) \tag{170}$$

Since $Sp(B)$ is a bounded set, δ can always be so chosen.

The optimum value of δ will be that which minimizes

$$(\delta) = \sup_{\lambda \in Sp(N_0^{-1}P_0)} \left|\frac{\lambda + \delta}{\delta + 1}\right| \tag{171}$$

Let

$$\left.\begin{aligned} \tau^- &= \inf_{\lambda \in Sp(N_0^{-1}P_0)} \lambda \\ \tau^+ &= \sup_{\lambda \in Sp(N_0^{-1}P_0)} \lambda < 1 \end{aligned}\right\} \tag{172}$$

Then

$$f(\delta) = \max\left\{\left|\frac{\tau^+ + \delta}{\delta + 1}\right|, \left|\frac{\tau^- + \delta}{\delta + 1}\right|\right\} \tag{173}$$

The graph in Fig. 2 shows that $f(\delta)$ is minimized if

$$\delta = -\frac{\tau^+ + \tau^-}{2} \tag{174}$$

Then the minimum is

$$f^* = \frac{|\tau^+ - \tau^-|}{|2 - \tau^+ - \tau^-|} \tag{175}$$

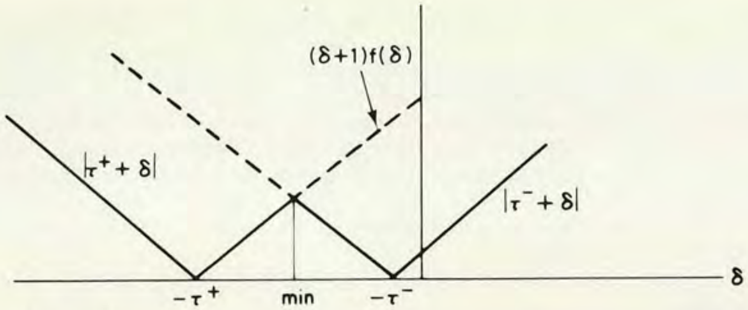

Figure 2.

and clearly for the value of δ given by Eq. (174) the process is accelerative in the sense that

$$\lim_{n\to\infty} \frac{\|r_{n,\delta}\|}{\|r_{n,0}\|} = 0 \tag{176}$$

since

$$f^* > \max(|\tau^+|, |\tau^-|) = f(\delta)|_{\delta=0} \tag{177}$$

which is the spectral radius of $(N_0^{-1}P_0 + \delta I)/(\delta + 1)$ when $\delta = 0$.

For the application of this method to the solution of systems of linear equations, the reader should consult Ref. 27, where methods of choosing N_0 are discussed.

The above methods in their philosophy are properly subsumed under the subject of iterative methods for solving equations, such as the methods of functional iteration and the Newton–Raphson method and its generalizations. A discussion of these methods is outside the scope of this article, and the reader is referred to Traub [29], Ostrowski [30], Householder [31] and Kantorovich and Akilov [32] (for a formulation for abstract spaces) for information about this subject.

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ACCESS AND ACCESSING

INTRODUCTION

Access, as used in the computer field, may be defined as the "ability to obtain or make use of" (1) data stored in machine-readable form; (2) the facilities of a computer system [1].

Included in the first implication is the reciprocal ability of storing data through access to a storage medium. This concept is central to the operation of the digital computer and is so extensively used that the term "access" has acquired usage as a

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PREDICTOR–CORRECTOR FORMULAS BASED ON RATIONAL INTERPOLANTS*

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Abstract—Title formulas are developed for particular use in the solution of nonlinear differential equations which in general offer no clue as to the presence of singularities on or near the path of integration. Procedure is advantageous since the approximations ascertain the existence of zeros and poles and locate these data with great accuracy. The function $y = J_1(x)/J_0(x)$ where $J_n(x)$ is the Bessel function of the first kind satisfies a first order nonlinear differential equation of the Riccati type, and has an infinite number of zeros and poles on the positive real axis. A numerical example is provided to illustrate computation of these singular points in $0 < x < 100$. Some other examples are also given.

1. SUMMARY AND INTRODUCTION

In the numerical treatment of ordinary linear differential equations there are many effective integration formulas which yield quick, reliable results. These techniques, which are based on the use of polynomial interpolating functions, depend heavily on the fact that solutions to linear differential equations are very well behaved in that all the needed higher order derivatives exist in the range of interest.

In general, the theory of ordinary nonlinear differential equations offers no clue as to the singularities of the solutions of such equations. Thus, the detection of singularities must be accomplished heuristically. Obviously the usual numerical integration techniques fail in the region of such singularity, but also the location of such a point evades detection. Hence, new techniques must be developed which will deal effectively with the problem of singularities of solutions to nonlinear differential equations. Progress has been made in this direction [1–4].

Recently we obtained an algorithm for computing rational approximations to the solution of a wide class of nonlinear differential equations[1]. Although this technique cannot be extended to arbitrary nonlinear equations, it clearly demonstrates the power of rational approximations in dealing with a function which has singular points in the range of interest. These approximations ascertain the existence of zeros and poles of a function and locate these critical points with great accuracy. Rational approximations also allow accurate computation of the function near a singular point—a decided advantage over the usual approximations. Thus, it is desirable to obtain formulas for numerical integration of ordinary nonlinear differential equations which are based on rational approximations. Historically, Lambert and Shaw[2, 3] were the first writers to treat quadrature formulas which are based on rational interpolants. In Ref. [2], these authors discussed a two-point formula for integrating the scalar differential equation.

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$$y' = f(x, y), \qquad y(x_0) = y_0 \tag{1}$$

which was based on an interpolant of the form

$$P_n(x)/(b + x) \tag{2}$$

P_n being an nth degree polynomial, as well as a three-point implicit formula based on the same interpolant. In Ref. [3], they treated two-point quadrature formulas based on the interpolants

$$P_n(x) + a|b + x|^N, \qquad P_n(x) + a|b + x|^N \ln |b + x|. \tag{3}$$

These formulas involved numerical values of the higher derivatives of f. In a later paper, Shaw[4] extended the results to multistep methods based on equation (3) in which values of the higher derivatives are not required. However, determination of the type and location of the singular points requires the solution of two simultaneous transcendental equations. Further, the singularities are restricted to those characterized by equation (3).

In this paper we discuss multistep formulas for integrating equation (1) which are of predictor–corrector type. They involve numerical values of y and f only, and are based on the general interpolating rational function

$$P_m(x)/Q_n(x). \tag{4}$$

Thus we abandon the possibility of characterizing the type of singularity specifically except in the case of multiple poles. In many applications, the nature of the singularity, if any, is not known *a priori*. This being the case, our approach is valuable as it is easy to apply and ably serves to alert one to the existence of singularities other than poles. In the proposed system, location of the singularity is obtained directly by computing zeros of the denominator polynomial.

2. DEVELOPMENT OF INTEGRATION FORMULAS

Here we develop multi-step predictor and corrector formulas based on rational functions for the numerical integration of the scalar differential equation

$$y' = f(x, y), y(x_0) = y_0. \tag{5}$$

Although the formulas can be developed for higher order equations, we restrict ourselves to equation (5).

It is assumed that all the needed starting values are available. We wish to approximate the solution to equation (5) by a rational function. We write

$$y(x) = \frac{P_m(x)}{Q_n(x)} + E_{m,n}(x)$$

$$P_m(x) = \sum_{j=0}^{m} a_j x^j, Q_n(x) = 1 + \sum_{j=1}^{n} b_j x^j, \tag{6}$$

where the coefficients a_j and b_j are unknown.

In order to obtain a k-step method the development that follows shows that it is sufficient to require that either $n + m = 2k$, or $n + m = 2k - 1$. For convenience, we set

$$L_{m,n}(x) = Q_n(x)y(x) - P_m(x), \tag{7}$$

$$L'_{m,n}(x) = Q'_n(x)y(x) + Q_n(x)y'(x) - P'_m(x), \tag{8}$$

and $y(x_j) = y_j$ for a set of equally spaced interpolation points $x_0 < x_1 < \ldots < x_{k+1}$ with spacing $h = x_{j+1} - x_j$. We develop the predictor and corrector formulas for the cases $m + n = 2k$ and $m + n = 2k - 1$. The pertinent equations are:

$$m + n = 2k, \qquad L_{m,n}(x_i) = 0 \qquad i = 0, 1, \ldots, k + 1$$
$$L'_{m,n}(x_i) = 0 \qquad i = 1, 2, \ldots, k \tag{9}$$

for the predictor and

$$L_{m,n}(x_i) = 0 \qquad i = 1, 2, \ldots, k + 1$$
$$L'_{m,n}(x_i) = 0 \qquad i = 1, 2, \ldots, k + 1 \tag{10}$$

for the corrector;

$$m + n = 2k - 1, \qquad L_{m,n}(x_i) = 0 \qquad i = 0, 1, \ldots, k$$
$$L'_{m,n}(x_i) = 0 \qquad i = 0, 1, \ldots, k - 1 \tag{11}$$

for the predictor and

$$L_{m,n}(x_i) = 0 \qquad i = 0, 1, \ldots, k$$
$$L'_{m,n}(x_i) = 0 \qquad i = 1, 2, \ldots, k \tag{12}$$

for the corrector.

We quote the formulas for the case $m + n = 2k$. Computations are greatly simplified if we employ the transformation $x = x_0 + th$ in equations (5)–(12), develop the required formulas and then set $t = (x - x_0)/h$. This procedure is equivalent to setting $x_0 = 0$, $x_j = j$ and, after the formulas are obtained, replacing y'_j by hy'_j.

The system of equations (9) in the unknowns a_j and b_j has a solution, if and only if, the following determinant vanishes;

$$\Delta_p = \begin{vmatrix} R_{k+2,m+1} & S_{k+2,n+1} \\ T_{k,m+1} & U_{k,n+1} \end{vmatrix} = 0, \tag{13}$$

where the entries in the determinant Δ_p are rectangular arrays of the size indicated by the subscripts, and if h_{ij} denotes the i, jth entry in a rectangular array H_{pq}, $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$, the arrays in the determinant (13) are defined by

$$r_{ij} = (i - 1)^{j-1}, \qquad i = 1, 2, \ldots, k + 2; j = 1, 2, \ldots, m + 1 \text{ and } r_{11} = 1,$$
$$s_{ij} = (i - 1)^{j-1} y_{i-1}, \qquad i = 1, 2, \ldots, k + 2; j = 1, 2, \ldots, n + 1 \text{ and } s_{11} = y_0,$$
$$t_{ij} = (j - 1)(i)^{j-2}, \qquad i = 1, 2, \ldots, k; j = 1, 2, \ldots, m + 1$$
$$u_{ij} = (j - 1)(i)^{j-2} y_i + (i)^{j-1} y'_i, \qquad i = 1, 2, \ldots, k; j = 1, 2, \ldots, n + 1. \tag{14}$$

We call Δ_p the predictor determinant. To obtain the corrector determinant, Δ_c, corresponding to the conditions (9), replace the first row of equation (13) by the last row of equation (13) with k replaced by $k + 1$. The modification for the case $m + n = 2k - 1$ is carried out in a similar way. After setting $\Delta_p = \Delta_c = 0$ and expanding the determinants, we obtain

$$\Delta_p = Ay_{k+1} + B = 0, \tag{15}$$

$$\Delta_c = Cy_{k+1}^2 + Dy_{k+1} + E = 0, \tag{16}$$

where the coefficients in equation (15) depend on y_j and y'_j, $j = 0, 1, \ldots, k$, and the coefficients in equation (16) depend not only on these values, but also on y'_{k+1}. Thus, to get the predictor formula, we replace y'_j by hy'_j in equation (15) and solve for y_{k+1}. In practice the predicted value of y_{k+1} is employed in the differential equation to get a predicted value of y'_{k+1}. Then the corector is used repeatedly until the results stabilize. However, for sufficiently small h, the predictor can be used alone.

We give three examples.

$m = n = 1$: The predicted value of y_2 is

$$y_2 = \frac{2y_0y_1 - 2y_1^2 + hy_0y'_1}{2y_0 - 2y_1 + hy'_1}, \tag{17}$$

and the corresponding corrector formula is

$$y_2^2 - 2y_1y_2 + (y_1^2 - h^2y'_1y'_2) = 0$$

which agrees with that of Lambert and Shaw[2].

$m = 1, n = 2$: The predictor is

$$y_2 = \frac{y_0^2(3y_1 + hy'_1) + y_1^2(2hy'_0 - 3y_0)}{y_0(4y_0 - 5y_1 + hy'_1) + y_1(y_1 + 2hy'_0) - 2h^2y'_0y'_1},$$

and for the corrector, the values in equation (16) with $k = 1$ are given by

$$\begin{aligned} C &= 3y_1 - 4y_0 - hy'_1 \\ D &= 5y_0y_1 - 3y_1^2 + hy_0y'_1 \\ E &= -y_0y_1^2 + 2h(y_0 - y_1)y_1y'_2 + 2h^2y_0y'_1y'_2. \end{aligned} \tag{18}$$

$m = n = 2$; The predictor is

$$y_3 = y_2 + \frac{\Delta y_0(\Delta y_1)^2 + (\Delta y_1)^3 - 4(\Delta y_0 + \Delta y_1)h^2y'_1y'_2 + 6h\Delta y_0\Delta y_1y'_2}{15\Delta y_0\Delta y_1 - (\Delta y_1)^2 - 6hy'_1(\Delta y_0 + \Delta y_1) + 4h^2y'_1y'_2 - 6h\Delta y_0y'_2},$$

and for the corrector, the values in equation (16) are given by

$$\begin{aligned} C &= 4\Delta y_1 - hy'_2 - 4hy'_1 \\ D &= -2Cy_2 + 4(\Delta y_1)^2 - 2h\Delta y_1y'_2 \\ E &= Cy_2^2 - y_2\Delta y_1(4\Delta y_1 - 2hy'_2) - 4hy'_3(\Delta y_1)^2 - hy'_2(\Delta y_1)^2 + 4h^3y'_1y'_2y'_3. \end{aligned} \tag{19}$$

3. THE TRUNCATION ERROR OF THE PREDICTOR FORMULAE

We write

$$y(x) = \frac{P_m(x)}{Q_n(x)} + \frac{L_{m,n}(x)}{Q_n(x)}. \tag{20}$$

Let (a, b) be some interval containing all the interpolation points in question. We assume that $f(x)$ can be written in the form

$$y(x) = \frac{g(x)}{\psi(x)}, \qquad \psi(x) = \prod_{s=1}^{r} (x - y_s)^{\nu_s}, \qquad \nu_s \text{ an integer} \geq 0, \tag{21}$$

$$\sum_{s=1}^{r} \nu_s \leq n, \qquad y_s \in (a, b), y_s \neq x_j,$$

where $g(x)$ is $(2k + 1)$ times differentiable for $x \in (a, b)$.

We further define

$$\Pi(x) = \prod_{j=1}^{k} (x - x_j)^2 \tag{22}$$

Case I, $m + n = 2k$:

Let

$$L_{m,n}(x) = \frac{(x - x_0)\Pi(x)A(x)}{\psi(x)} \tag{23}$$

where $A(x)$ is a function to be determined. If we define

$$\omega(t) = f(t)Q_n(t)\psi(t) - P_m(t)\psi(t) - A(x)(t - x_0)\Pi(t) \tag{24}$$

where $x \neq x_j, x \in (a, b)$, we see that $\omega(t)$ vanishes if t is one of the points $(x_0, x_1, \ldots, x_{r-1}, x, x_r, \ldots, x_k)$, since

$$\omega(t) = \psi(t)L_{m,n}(t) - A(x)(t - x_0)\Pi(t). \tag{25}$$

This implies that $\omega'(t)$ must vanish for the $k + 1$ distinct points η_j, where $x_0 < \eta_1 < x_1$, $x_1 < \eta_2 < x_2, \ldots, x_{r-1} < \eta_r < x, x < \eta_{r+1} < x_r, x_{k-1} < \eta_{k+1} < x_k$, by Rolle's theorem. Also $\omega'(t)$ vanishes at the points $x_1, \ldots, x_k$, as may be seen by differentiating equation (25) and using equation (9). Altogether, $\omega'(t)$ vanishes for at least $2k + 1$ distinct points in the interval (a, b). Thus, $\omega''(t)$ vanishes for at least $2k$ distinct points, etc. Finally, $\omega^{(2k+1)}(t)$ must vanish for at least one point, say ξ, $\xi \in (a, b)$. If we differentiate equation (24) $(2k + 1)$ times, the second term on the right hand side vanishes, since degree $(P_m\psi) \leq n + m = 2k$. Placing $t = \xi$ in the remaining terms gives

$$A(x) = \frac{1}{(2k+1)!} \frac{\mathrm{d}^{2k+1}}{\mathrm{d}t^{2k+1}} [f(t)Q_n(t)\psi(t)]_{t=\xi}, \quad \xi \in (a, b),$$

so from equation (23) we see

$$y(x) - \frac{P_m(x)}{Q_n(x)} = \frac{(x - x_0)\Pi(x)A(x)}{\psi(x)Q_n(x_{k+1})}.$$

Putting $x = x_{k+1}, x_r = x_0 + rh$, we find

$$y_{k+1}(\text{true}) - y_{k+1}(\text{predicted}) = \frac{(k + 1)k!^2h^{2k+1}}{(2k + 1)!\psi(x_{k+1})Q_n(x_{k+1})} \frac{\mathrm{d}^{2k+1}[f(t)Q_n(t)\psi(t)]_{t=\xi}}{\mathrm{d}t^2{}_{k+1}}$$

for some $\xi \in (a, b)$, where (a, b) is any interval containing the points $x_0, x_1, \ldots, x_n, x_{k+1}$.

Case II, $m + n = 2k - 1$:

A similar analysis yields

$$y_{k+1}(\text{true}) - y_{k+1}(\text{predicted}) = \frac{k!^2\, h^{2k}}{(2k)!\psi(x_{k+1})Q_n(x_{k+1})} \frac{\mathrm{d}^{2k}}{\mathrm{d}t^{2k}} [f(t)Q_n(t)\psi(t)]_{t=\xi}$$

for some $\xi \in (c, d)$, where (c, d) is any interval containing the points $(x_1, x_2, \ldots, x_k, x_{k+1})$.

The truncation error analysis for the corrector formulas seems to be a much more difficult undertaking, and our studies of the error for these formulas are still continuing. It is a fairly easy matter to show that the truncation error associated with the formulas (10) and (12) is at least $0(h^{2k})$ and $0(h^{2k-1})$ respectively, but numerical evidence and heuristic arguments lead us to conjecture that the correcter formulas are as accurate as the predictor formulas, i.e. $0(h^{2k+1})$ and $0(h^{2k})$, respectively. We hope a more refined analysis will substantiate this conjecture.

4. EXAMPLES AND APPLICATIONS

We apply the results of Section 2 to four examples. In the first two examples, first order differential equations whose only singularities are poles are treated. Here the approximations with $m = 1$, $n = 2$ are employed. In the third example, solution of a second order differential equation with a single simple pole, is found using the $m = n = 2$ procedure. Finally, in the fourth example, we study the solution of a first order differential equation with an essential singularity using the schemata $m = n = 1$ and $m = 1$, $n = 2$. In all cases, the corrector is used repeatedly until successive iterates agree to eight decimal places. The root of equation (16) was chosen which agreed best with the previously computed value of y_{k+1}. In the interests of brevity, only portions of the tabular data are exhibited. We now discuss the examples in more detail.

Tables 1 and 2 show how the technique can be used to map a solution and locate poles. Indeed, it appears that the path of integration can go right through a pole without any serious adverse effects on the solution beyond the pole. (See later discussion surrounding Table 3.) Warning of the presence of a pole and its approximate location can be deduced by tabulating roots of $Q_2(x)$. A recommended procedure to get the pole is to tabulate C^{-1} vs x, C = Corrected, and inverse interpolate for a zero of C^{-1}.

Table 3 manifests the great power of our scheme. Here the numerical solution for the differential equation for $v = J_1(x)/J_0(x)$ was mapped using the $m = 1$, $n = 2$ procedure for $x = 0$ to $x = 99{\cdot}99$ with $h = 0{\cdot}01$.

Only portions of the calculation are exhibited. Both zeros and poles are readily deduced as indicated. Notice that the roots of $Q_2(x)$ indicate the presence of a pole, but this is not reliable for more precise location of the pole since for x near a pole and $x > 5{\cdot}0$, the two zeros of $Q_2(x)$ are near each other. We also observe that in the neighborhood of each pole, $P_1(x)$ is an approximate factor of $Q_2(x)$. There is good reason for this since from the differential equation for $v(x)$, $v(x) \sim -(x - x_0)^{-1}$ when x is near x_0 where x_0 is a pole of $v(x)$. In going from $x = 0$ to $x = 99{\cdot}99$, $v(x)$ passes through 32 poles and 31 zeros. The stability of the integration process is remarkable.

Table 4 illustrates numerical solution of a second order differential equation. Z is known to have a simple pole at $1{\cdot}1577$. In this example, no attempt was made to determine the pole as in the manner of the examples in Table 3.

Table 1. Numerical solution of $u' = 1 + u^2$, $u(0) = 1$

		$u = \tan(x + \pi/4)$ $m = 1, n = 2$ $h = 0.05$		$h = 0.01$	
x	True (u)	Predicted	Corrected	Predicted	Corrected
0·1	1·22305	1·22304	1·22305	1·22305	1·22305
0·2	1·50850	1·50848	1·50850	1·50850	1·50850
0·3	1·89577	1·89574	1·89577	1·89577	1·89577
0·4	2·46496	2·46493	2·46498	2·46496	2·46496
0·5	3·40822	3·40815	3·40826	3·40822	3·40822
0·6	5·33186	5·33165	5·33195	5·33186	5·33186
0·7	11.68137	11·67998	11·68153	11·68138	11·68139
0·8	−68·47967	−68·59667	−68·66273	−68·48685	−68·49443
0·9	−8·68763	−8·73393	−8·68629	−8·69860	−8·69493
1·0	−4·58804	−4·62137	−4·64804	−4·56120	−4·56121

Finally Table 5 shows the results for the case of an essential singularity. In the $m = n = 1$ data, the root of $Q_1(x)$ is indicating the proximity of a singular point. When $m = 1$, and $n = 2$, as x nears the singular point we should expect $Q_2(x)$ to have nearly equal roots since the best a rational approximation can do to mimic an essential singularity is to display a second order pole. However, the data show that $P_1(x)$ is nearly a factor of $Q_2(x)$, so that the rational interpolant behaves like a simple pole. The following is an explanation of this behavior. By integration of the differential equation and application of the mean value theorem, we have

$$y(x) = 0{\cdot}2\int_0^x \frac{y(t)\,\mathrm{d}t}{(1-t)^2} = \frac{0{\cdot}2xy(\xi)}{1-x}, \qquad 0 < \xi < x < 1.$$

In any event, the $m = 1, n = 2$ results are quite good in the neighborhood of the singularity.

5. CONCLUSION

It is well-known that rational approximations are extremely effective for numerical computation, particularly when the function to be approximated has singularities in the nature of branch points and poles. Rational approximations also supply a means of locating

Table 2. Location of pole of tan $(x + \pi/4)$; true pole; $\pi/4 = 0{\cdot}785398$

	$h = 0{\cdot}05$		$h = 0{\cdot}01$
x	Root of $Q_2(x)$	x	Root of $Q_2(x)$
0·60	0·7868	0·76	0·78540 35
0·65	0·7855	0·77	0·78539 82
0·70	0·7869	0·78	0·78540 08
0·75	0·7851	0·79	0·78539 89
0·80	0·7851	0·80	0·78539 65

By inverse interpolation of C^{-1}, the pole is at 0·7851 for the $h = 0{\cdot}05$ case and at 0·785398 for the $h = 0{\cdot}01$ case.

these singularities. The algorithm described in this paper for integration formulas based on rational interpolants is straight forward, easily adapted to computer use and possesses all the advantage of rational approximations.

Table 3. Numerical solution of $v' = 1 + v^2 - v/x$, $v(0) = 0$

$v = J_1(x)/J_0(x)$

$m = 1, n = 2, h = 0{\cdot}01$

x	True $y(x)$	Predicted	Corrected	Root of $P_1(x)$	Roots of $Q_2(x)$
2.38	40·48251	40·48279	40·48277	3·00768	3·09490, 2·40483
2.39	67·65514	67·65590	67·65586	3·03908	3·13574, 2·40483
2·40	207·43659	207·44372	207·44358	3·07192	3·17936, 2·40482
2.41	−193·04827	−193·04220	−193·04260	3·11165	3·23341, 2·40483
2·42	−65·68773	−65·68773	−65·68763	3·15420	3·29302, 2·40482

From the roots of $Q_2(x)$, an approximate pole is at 2·40482. By inverse interpolation of C^{-1}, a pole is indicated at 2·40483 which is correct to five decimals.

	True $y(x)$	Predicted	Corrected	Root of $P_1(x)$	Roots of $Q_2(x)$
3·82	−0·01172	−0·01172	−0·01172	3·83171	5·71681, 2·31875
3.83	−0·00171	−0·00171	−0·00171	3·83171	5·71727, 2·31847
3·84	0·00829	0·00829	0·00829	3·83171	5·71759, 2·31826
3·85	0·01825	0·01825	0·01825	3·83171	5·71780, 2·31813
3.86	0·02820	0·02820	0·02820	3·83171	5·71788, 2·31807

Thus a zero is at 3·83171 which is correct to five decimals.

x	Predicted	Corrected	Root of $P_1(x)$	Roots of $Q_2(x)$
98·16	−0·01260	−0·01260	98·17260	99·91198, 96·44848
98·17	−0·00260	−0·00260	98·17260	99·91217, 96·44831
98·18	0·00740	0·00740	98·17260	99·91224, 96·44824
98·19	0·01740	0·01740	98·17260	99·91219, 96·44829
98·20	0·02740	0·02740	98·17260	99·91204, 96·44845

Thus a zero is indicated at 98·17260. The true zero is at 98·17095.

x	Predicted	Corrected	Root of $P_1(x)$	Roots of $Q_2(x)$
99·73	54.14439	54.14470	99·6734	99·78237, 99·63947
99·74	118·08285	118·08425	99·71568	99·80917, 99·65500
99·75	−653·13693	−653·16633	99·74525	99·74757 ± 0·24049i
99·76	−86·72376	−86·73472	99·75690	99·82040, 99·68518
99·77	−46·44744	−46·43145	99·76146	99·75496 ± 0·01452i

By inverse interpolation of C^{-1}, a pole is indicated at 99·74847. The true pole is at 99·74682.

Table 4. Numerical solution of $Z'' = 2Z^3 + xZ + 1$, $Z(0) = 1$, $Z'(0) = 0$ (Z is Painleve's second transcendent)

		$m = n = 2$			
		$h = 0{\cdot}05$		$h = 0{\cdot}01$	
x	True (Z)	Predicted	Corrected	Predicted	Corrected
0·2	1·06261	1·06271	1·06267	1·06261	1·06261
0·3	1·14638	1·14634	1·14640	1·14637	1·14638
0·4	1·27415	1·27377	1·27379	1·27415	1·27415
0·5	1·45921	1·45751	1·45730	1·45921	1·45921
0·6	1·72538	1·72170	1·72159	1·72537	1·72538
0·7	2·11844	2·11211	2·11185	2·11840	2·11845
0·8	2·73694	2·72608	2·72581	2·73710	2·73710
0·9	3·83440	3·81512	3·81417	3·83522	3·83520
1·0	6·31100	6·24525	6·25787	6·31763	6·31758
1·1	17·31546	19·30070	21·69210	17·37845	17·38471

Table 5. Numerical solution of $y'(x) = [0{\cdot}2y/(1 - x)^2]$, $y(0) = e^{0\cdot 2}$

$y(x) = \exp(0{\cdot}2/1 - x)$

$m = n = 1$ $h = 0{\cdot}02$

x	True $y(x)$	Predicted	Corrected	Roots of $Q_1(x)$
0·84	3·49034	3·49158	3·48947	1·06828
0·86	4·17273	4·17601	4·17070	1·06501
0·88	5·29449	5·30474	5·28895	1·06117
0·90	7·38906	7·42994	7·37009	1·05657
0·92	12·18249	12·42287	12·08980	1·05091

$m = 1, n = 2, h = 0{\cdot}01$

x	True $y(x)$	Corrected	Root of $P_1(x)$	Roots of $Q_2(x)$
0·85	3·79367	3·79363	25·13805	0·94305, 0·22946
0·86	4·17273	4·17268	−0·20523	0·94439, 0·47314
0·87	4·65742	4·65732	0·46970	0·94576, 0·61113
0·88	5·29449	5·29429	0·68748	0·94716, 0·71860
0·89	6·16065	6·16016	0·79565	0·94851, 0·80040
0·90	7·38906	7·38753	0·86088	0·94969, 0·86107

$m = 1, n = 2, h = 0{\cdot}005$

x	True $y(x)$	Corrected	Root of $P_1(x)$	Roots of $Q_2(x)$
0·90	7·38906	7·38885	0·88453	0·94992, 0·88454
0·91	9·22781	9·22789	0·91452	0·94998, 0·91452
0·92	12·18249	12·23246	0·99119	$0{\cdot}96280 \pm 0{\cdot}00761i$
0·93	17·41171	17·48333	0·99909	$0{\cdot}96312 \pm 0{\cdot}01005i$
0·94	28·03162	28·14705	1·02628	$0{\cdot}96676 \pm 0{\cdot}00900i$
0·95	54·59815	54·82113	1·09941	$0{\cdot}97043 \pm 0{\cdot}00734i$

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Some Transformations of Monotone Sequences

By Jet Wimp

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Some Transformations of Monotone Sequences*

By Jet Wimp

Abstract. In this paper, we discuss a class of methods for summing sequences which are generalizations of a method due to Salzer. The methods are not regular, and in contrast to the classical regular methods, seem to work best on sequences which are monotone. In our main theorem, we determine a class of convergent sequences for which the methods yield sequences which converge to the same sum.

1. Introduction. In this paper, we discuss a class of transformations which are useful for summing certain monotone sequences.

In what follows, let k, m, n be integers, $k \geqq 0$, $m, n \geqq 1$, let $\{S_n\}$ be a sequence of complex numbers and λ be a complex number. The transformation $\mathfrak{U}_m$: $\{S_n\} \to \{S_n^*\}$ is defined by

$$S_m^* = \mathfrak{U}_m(\{S_n\}) = \mathfrak{J}_m(\{S_n\})/\mathfrak{J}_m(\{1\}), \tag{1}$$

where

$$\mathfrak{J}_m(\{S_n\}) \equiv \mathfrak{J}_m(k, \lambda, \{S_n\}) = \frac{1}{m!} \sum_{r=1}^{m} (-)^{r+1}(\lambda + r)^{k+m}\binom{m}{r} S_r. \tag{2}$$

$\mathfrak{U}_m$, for the case where $k = 0$, $\lambda = -N$, was discussed by Salzer [1], [2], [3]. He was interested in using $\mathfrak{U}_m$ as a summation process for converting the slowly convergent (or divergent) sequence $\{S_n\}$ into a more rapidly convergent (or convergent) sequence $\{S_n^*\}$. Although Salzer provided no convergence criteria, he did furnish a number of practical examples where $S_n^* \to \alpha$ when $S_n \to \alpha$. This is not, however, true in general, as is shown in Section 2. In Section 3, we describe a class of sequences for which S_n^* is convergent, and we close with an example.

2. The Regularity of $\mathfrak{U}_m$. $\mathfrak{U}_m$ is not regular. To show this, we require a

LEMMA.

$$\begin{aligned} \mathfrak{J}_m(\{1\}) &= \frac{1}{2\pi i} \int_{\Gamma_m} \frac{(\lambda + z)^{k+m}}{(-z)_{m+1}}\, dz \\ &= \frac{\lambda^{k+m}}{m!} + \frac{(-)^{m+1}}{m!} (k + 1)_m B_k^{(-m)}(\lambda) \qquad (3) \\ &= A(-)^{m+1} m^{2k}[1 + O(m^{-1})], \qquad m \to \infty,\ A \neq 0, \end{aligned}$$

where Γ_m is a simple closed curve encircling the integers $1, 2, \cdots, m$, $\operatorname{Re} z > 0$, $z \in \Gamma_m$, *and* $(\mu)_k = \mu(\mu + 1) \cdots (\mu + k - 1)$.

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Proof. Evaluate the integral above by residues. This gives (2) with $S_r \equiv 1$. Now use the formula

$$(\lambda + r)^{k+m} = \sum_{s=0}^{k+m} \binom{k+m}{s} (-)^s (-r)_s B_{k+m-s}^{(-s)}(\lambda), \tag{4}$$

(Nörlund [4, p. 150]), substitute this in (2) and interchange the order of summation. The lemma follows immediately if we use the fact that the Bernoulli polynomial $B_k^{(-m)}(\lambda)$ is a polynomial in m of degree k.

To show $\mathfrak{U}_m$ is not regular, let $\{S_n\}$ be the null sequence

$$S_n = (-)^{n+1}/(n + \lambda), \tag{5}$$

and suppose $\lambda > 0$. Then

$$\begin{aligned} |\mathfrak{Z}_m(\{S_n\})| &= \frac{1}{m!} \sum_{r=1}^{m} (\lambda + r)^{k+m-1} \binom{m}{r} \\ &\geq \left(\lambda + \frac{m}{2}\right)^{k+m-1} \Big/ \Gamma\left(\frac{m}{2} + 1\right)^2 \\ &> Ce^m m^k, \quad \text{for } m \text{ even and some } C > 0, \end{aligned} \tag{6}$$

so, by using (3), we see that

$$S_m^* > De^m m^{-k}, \qquad m \text{ even and for some } D > 0, \tag{7}$$

and S_m^* does not converge.

3. A Class of Sequences Summed by $\mathfrak{U}_m$. A distinguishing characteristic of the transformation $\mathfrak{U}_m$ is, generally speaking, that it sums best (in the sense that if $S_n \to \alpha$, then $S_n^* \to \alpha$ more rapidly) those sequences which are monotone, and is less effective on summing sequences which are not monotone. Exactly the opposite is true of most of the classical summation procedures, such as the Cesaro summability method, which work best on those sequences the successive differences of whose members alternate in sign.

In the following theorem, we explore this interesting feature of $\mathfrak{U}_m$ by determining a class of sequences for which $S_m^* \to \alpha$ if $S_m \to \alpha$.

Let in what follows

$$S_n = \alpha + R_n, \qquad S_n^* = \alpha + R_n^*. \tag{8}$$

THEOREM. *Let there exist a function $R(z)$ analytic for* Re $z \geq p$ *for some p, $0 < p < 1$, such that*

$$R(z) = O(|z|^\mu) \quad \textit{for some} \quad \mu < -k, \tag{9}$$

as $|z| \to \infty$, Re $z \geq p$. *Further, let*

$$R(n) = R_n, \qquad n \geq 1. \tag{10}$$

Then

$$\mathfrak{U}_m(\{S_n\}) = \alpha + o(m^{-2k}), \qquad m \to \infty. \tag{11}$$

Proof. Under the given conditions, we have

$$\alpha - \mathfrak{U}_m(\{S_n\}) = R_m^* = \frac{\mathfrak{J}_m(\{1\})^{-1}}{2\pi i} \int_{\Gamma_m} \frac{(\lambda + z)^{k+m}}{(-z)_{m+1}} R(z)\, dz \tag{12}$$
$$= \frac{\mathfrak{J}_m(\{1\})^{-1}}{2\pi i} \int_{p-i\infty}^{p+i\infty} \frac{(\lambda + z)^{k+m}}{(-z)_{m+1}} R(z)\, dz,$$

so

$$|R_m^*| \leqq Cm^{-2k} \int_{-\infty}^{\infty} |p + iu|^{\mu-1}\, |p + \lambda + iu|^k \prod_{j=1}^{m} \frac{|p + \lambda + iu|}{|p - j + iu|}\, du. \tag{13}$$

We can choose constants A and $\epsilon > 0$ such that

$$|p + iu|^{\mu-1}\, |p + \lambda + iu|^k \leqq A(|u| + \epsilon)^{\mu+k-1}, \qquad -\infty < u < \infty. \tag{14}$$

Then, we have

$$|R_m^*| \leqq C'm^{-2k} \int_0^{\infty} (u + \epsilon)^{\mu+k-1} \prod_{j=1}^{m} \frac{[(\lambda + p)^2 + u^2]^{1/2}}{[(j - p)^2 + u^2]^{1/2}}\, du. \tag{15}$$

Now, we may write

$$\frac{(\lambda + p)^2 + u^2}{(j - p)^2 + u^2} = 1 + \frac{(\lambda + p)^2 - (j - p)^2}{(j - p)^2 + u^2}, \tag{16}$$

so it is clear we may choose j_0, $1 \leqq j_0 \leqq m$ so that the left-hand side above is monotone increasing in u for $j > j_0$ and monotone decreasing for $j \leqq j_0$ (in fact, $j_0 = \sup\{\text{integral part } (\lambda + 2p), 1\}$).

We have

$$|R_m^*| \leqq C''m^{-2k}\left\{\int_0^{m^{1/2}} + \int_{m^{1/2}}^{\infty} (u + \epsilon)^{\mu+k-1} \prod_{j=j_0+1}^{m} \frac{[(\lambda + p)^2 + u^2]^{1/2}}{[(j - p)^2 + u^2]^{1/2}}\, du\right\} \tag{17}$$

$$\leqq C''m^{-2k}\left\{K \prod_{j=j_0+1}^{m} \frac{[(\lambda + p)^2 + m]^{1/2}}{[(j - p)^2 + m]^{1/2}} + \int_{m^{1/2}}^{\infty} (u + \epsilon)^{\mu+k-1}\, du\right\} \tag{18}$$

$$\leqq C''m^{-2k}\left\{K \frac{[(\lambda + p)^2 + m]^{m/2}}{(j_0 + 1 - p)_{m-j_0}} + \frac{(m^{1/2} + \epsilon)^{\mu+k}}{(\mu + k)}\right\} \tag{19}$$

$$= o(1)m^{-2k}. \tag{20}$$

Equation (20) follows from (19) when Stirling's formula is used. This completes the proof.

As an example, let

$$S_n = \sum_{k=1}^{n} [k(\omega + k)]^{-1}, \qquad \omega \neq -1, -2, -3, \cdots, \tag{21}$$

so that

$$\alpha = (\psi(\omega) + \gamma + \omega^{-1})\omega^{-1}, \tag{22}$$

$$R_n = (\psi(n + 1) - \psi(\omega + n + 1))\omega^{-1}. \tag{23}$$

We can let

$$R(z) = (\psi(z + 1) - \psi(\omega + z + 1))\omega^{-1} = -z^{-1} + O(z^{-2}), \tag{24}$$

arg $z < \pi$. Thus, the theorem shows that for $k = 0$, $S_m^* \to \alpha$. Unfortunately, the theorem is far too conservative. It provides an estimate of $o(1)$ for R_m^*, while numerical evidence shows that R_m^* goes to zero much more rapidly than R_m, which is $O(m^{-1})$ as $m \to \infty$.

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The Summation of Series Whose Terms Have Asymptotic Representations

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I. INTRODUCTION

Previous writers have discussed the expansion of the Meijer G-function

$$f(x) = G_{p,q}^{m,k}\left(x \,\middle|\, {}^{a_p}_{b_q}\right) \tag{1.1}$$

in series of the classical orthogonal polynomials of argument λ/x; see [1, 2]. (For the theory of the special functions used in this paper, the Erdélyi volumes [3] provide a good reference.) Without giving conditions, we write the pertinent expansions in the form

$$f(x) = \sum_{n=0}^{\infty} C_n(\lambda)\, R_n^{(\alpha,\beta)}\left(\frac{\lambda}{x}\right) \qquad 1 \leqslant x/\lambda \leqslant \infty \tag{1.2}$$

and

$$f(x) = \sum_{n=0}^{\infty} D_n(\lambda)\, L_n^{(\alpha)}\left(\frac{\lambda}{x}\right) \qquad 0 < \lambda/x < \infty. \tag{1.3}$$

We have used the notation $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1)$ for the shifted Jacobi polynomial. It has also been shown [4, 5] that for particularly important values of m, k, p, and q the coefficients C_n, D_n may be represented asymptotically by

$$\left.\begin{aligned} C_n(\lambda) &= d_1 n^{\theta_1} e^{-a_1 n^{2/\rho}}\,[1 + O(n^{-\delta_1})], \qquad n \to \infty \\ D_n(\lambda) &= d_2 n^{\theta_2} e^{-a_2 n^{2/\rho}}\,[1 + O(n^{-\delta_2})], \qquad n \to \infty \end{aligned}\right\} \tag{1.4}$$

where

$$\rho = q - p + 2 \geqslant 3.$$

Furthermore, in [6, 7], C_n, D_n are shown to satisfy certain difference equations whose coefficients are polynomials in n. In fact, these difference equations are of the so-called irregular type, first studied by writers such as Adams [8] and Batchelder [9]. The theory of these equations was completed by Birkhoff and Trjitzinsky in two of the most important and most overlooked papers of classical analysis, see [10] and [11]. Briefly, these authors showed that any function which satisfies a difference equation in ω whose coefficients possess convergent or even asymptotic expansions in powers of $\omega^{-1/r}$, r an integer, will possess an asymptotic expansion given by a linear combination of series of the form

$$y(\omega) \sim e^{Q(\omega)} \sum_{j=0}^{m} (\ln \omega)^j s_j(\omega), \qquad \omega \to \infty, \tag{1.5}$$

where

$$Q(\omega) = \mu_0 \omega \ln \omega + \mu_1 \omega + \mu_2 \omega^{(\rho-1)/\rho} + \mu_3 \omega^{(\rho-2)/\rho} + \cdots + \mu_\rho \omega^{1/\rho}, \tag{1.6}$$

$$s_j(\omega) = \omega^\theta [\alpha_{0,j} + \alpha_{1,j} \omega^{-1/\rho} + \alpha_{2,j} \omega^{-2/\rho} + \cdots], \tag{1.7}$$

and ρ is an integral multiple of r, μ_0 an integral multiple of $1/\rho$. In this context, the term "asymptotic expansion" has the following meaning. The functions $\{(\ln \omega)^s \, \omega^{-t/\rho}\}$ $t = 0, 1, 2,..., s = 0, 1, 2,..., m$, can be arranged in descending order of growth as $\omega \to \infty$. Let this arrangement be $\{x_0(\omega), x_1(\omega), x_2(\omega), x_3(\omega) \cdots\}$. The above expansion can be written

$$e^{Q(\omega)} \omega^\theta \sum_{j=0}^{\infty} x_j(\omega), \tag{1.8}$$

and what we mean by (1.5) is

$$e^{-Q(\omega)} \omega^{-\theta} y(\omega) - \sum_{j=0}^{r} x_j(\omega) = 0[x_{r+1}(\omega)], \; \omega \to \infty. \tag{1.9}$$

It is easy to show that such Birkhoff expansions, as we shall call them, are unique, like the ordinary Poincaré asymptotic series ($Q \equiv 0, \rho = 1$). The actual coefficients of the expansion are determined by substitution using such elementary identities as

$$(\omega + k)^\alpha = \omega^\alpha [1 + (\alpha k/\omega) + \cdots] \tag{1.10}$$

$$\begin{aligned} [\ln(\omega + k)]^s &= \left[\ln \omega + \ln \left(1 + \frac{k}{\omega}\right)\right]^s \\ &= (\ln \omega)^s + s(\ln \omega)^{s-1} \left(\frac{k}{\omega} - \frac{k^2}{2\omega^2} + \cdots\right) + \cdots \end{aligned} \tag{1.11}$$

and forcing the coefficients of $x_j(\omega)$ to vanish, see Birkhoff [10].

In view of these observations, we can see that the isolated facts about the Jacobi polynomial expansions begin to fit together. For instance, the steepest descent analyses in the references [4, 5] which yielded the equations (1.4) provided, in fact, the leading terms of the Birkhoff expansions for $C_n(\lambda)$ and $D_n(\lambda)$. Once these leading terms (or connecting constants for the expansions) are known, then the higher order terms can be obtained from the difference equation by the purely algebraic methods discussed above.

Now, an excellent way of computing $f(x)$ is to let $x = \lambda$ in (1.2) (the expansion (1.3) is in general much less rapidly convergent). Since

$$R_n^{(\alpha,\beta)}(1) = (\alpha + 1)_n/n! \tag{1.12}$$

we will then have a series which is rapidly convergent, which does not involve Jacobi polynomials, and whose terms possess Birkhoff expansions in n. Of course, many other series of this type are often encountered in practice.

Methods for summing such series that would take advantage of the fact that the n-th term possessed an asymptotic expansion of the Birkhoff type would be of great general interest; and in the first part of this paper we discuss such methods.

We shall see that these methods can be used to derive asymptotic expansions not only for the remainder of the series $\sum a_n$ where a_n has a Birkhoff representation, but also for the remainder of series of the form $\sum a_n P_n$, where a_n has a Birkhoff representation and P_n is a classical orthogonal polynomial. The reason for this is that P_n must satisfy a recursion relation of order three with polynomial coefficients, and hence itself may be represented as the sum of two Birkhoff series.

II. Applications to Ordinary Series

We first require the following.

THEOREM 1 (Birkhoff-Trjitzinsky). *Let the series*

$$A = \sum_{n=0}^{\infty} a_n \tag{2.1}$$

converge. Let

$$S_n = \sum_{r=0}^{n-1} a_r, \qquad R_n = A - S_n \tag{2.2}$$

and let

$$a_n = a(\omega) \sim e^{Q(\omega)} s(\omega), \qquad \omega = n + \zeta, \quad n \to \infty, \tag{2.3}$$

$Q(\omega)$ as above and

$$s(\omega) = \omega^{\theta}(\alpha_0 + \alpha_1\omega^{-1/\rho} + \alpha_2\omega^{-2/\rho} + \cdots). \tag{2.4}$$

Then R_n possesses the asymptotic representation

$$R_n = R(\omega) \sim e^{Q(\omega)}s^*(\omega), \qquad n \to \infty \tag{2.5}$$

where

$$s^*(\omega) = \omega^{\theta^*} \sum_{m=0}^{\infty} \beta_m \omega^{-m/\rho}. \tag{2.6}$$

Furthermore, θ^*, β_0 are as follows.

Case I: $Q \not\equiv 0$. Let the first nonzero μ_j in the sequence $[\mu_0, \mu_1, ..., \mu_\rho]$ be denoted by μ_τ. Then

$$\theta^* = \begin{cases} \theta, & \tau = 0 \\ \theta + (\tau - 1)/\rho, & 1 \leqslant \tau \leqslant \rho \end{cases} \tag{2.7}$$

$$\beta_0 = \begin{cases} -\alpha_0, & \tau = 0 \\ \alpha_0/(1 - e^{\mu_1}), & \tau = 1 \\ -\alpha_0\rho/[\mu_\tau(\rho + 1 - \tau)], & 2 \leqslant \tau \leqslant \rho \end{cases} \tag{2.8}$$

Case II: $Q \equiv 0$.

$$\theta^* = \theta + 1, \qquad \beta_0 = \alpha_0/(\theta + 1). \tag{2.9}$$

Proof. We have

$$R_{n+1} - R_n = -a_n \tag{2.10}$$

or

$$R(\omega + 1) - R(\omega) = -a(\omega), \qquad \omega = n + \alpha, \tag{2.11}$$

and the conclusion (2.5) follows directly from Lemma 8 in [11, p. 30]. There are, however, two points that require clarification. First, there is an arbitrary constant to be added to the series for $R(\omega)$, but the constant must be zero since $R(\omega) \to 0$. Next, no logarithms can occur in the series for $R(\omega)$, since logarithms will occur if and only if $Q \equiv 0$ and $s(\omega)$ contains a term ω^{-1}, see Birkhoff [10, p. 220]. However, such a case is excluded by the requirement that $\sum a_n$ converge. The actual determination of the constants θ^*, β_m in $s^*(\omega)$ is accomplished by using simple identities such as (1.10, 1.11), see Birkhoff [10] for examples.

We find, for instance, that

$$e^{\Delta Q(\omega)} = \omega^{\mu_0} e^{\mu_0+\mu_1}[1 + ((\rho + 1 - \sigma)/\rho)\,\mu_\sigma \omega^{(1-\sigma)/\rho} + O(\omega^{-\sigma/\rho})] \quad (2.12)$$
$$Q \not\equiv 0,$$

where μ_σ is the first nonzero μ_j in the sequence $\mu_2, \mu_3, \ldots, \mu_\rho$.

From the difference equation (2.11) it follows that we must have

$$\begin{aligned}
&\{\omega^{\mu_0} e^{\mu_0+\mu_1}[1 + ((\rho + 1 - \sigma/\rho)\,\mu_\sigma \omega^{(1-\sigma)/\rho} + O(\omega^{-\sigma/\rho})] - 1\} \\
&\qquad \times \{\beta_0 + \beta_1 \omega^{-1/\rho} + \cdots + \beta_\rho \omega^{-1} + O(\omega^{-1-1/\rho})\} \\
&\quad = -\omega^{\theta-\theta^*}\{\alpha_0 + O(\omega^{-1/\rho})\}. \qquad (2.13)
\end{aligned}$$

The requirement that the leading coefficient of both sides agree, leads to (2.7), (2.8). (Note that $\mu_0 \leqslant 0$).

If $Q \equiv 0$, the computations are even easier and we find

$$(\theta^*\beta_0/\omega) + (-(\beta_1/\rho) + \theta^*\beta_1)\,\omega^{-1-1/\rho} + \cdots = \omega^{\theta-\theta^*}(\alpha_0 + \alpha_1\omega^{-1/\rho} + \cdots) \quad (2.14)$$

from which follows (2.9).

THEOREM 2. *Let* (2.1), (2.2) *hold but let*

$$a(\omega) \sim e^{Q(\omega)}(\ln \omega)^p \sum_{m=0}^{\infty} \beta_m \omega^{-m/\rho} \quad (2.15)$$

p a positive integer. Then the asymptotic expansion of $R(\omega)$ may be obtained by formally differentiating the series $e^{Q(\omega)}s^*(\omega)$ p times with respect to θ. Similar asymptotic estimates for $R(\omega)$ may be obtained when $a(\omega)$ is any linear combination of series of the kind (2.15) *for different values of p.*

Proof. The above follows from the same Lemma in [11, p. 30].

To analyze the error, we introduce the following quantities

$$s_k^*(\omega) = \omega^{\theta^*} \sum_{m=0}^{k-1} \beta_m \omega^{-m/\rho}, \qquad k = 1, 2, 3, \ldots, \qquad \operatorname{Re} \omega > 0, \quad (2.16)$$

β_m, θ^* as above,

$$R(\omega) = e^{Q(\omega)}[s_k^*(\omega) + \epsilon_k(\omega)\omega^{-k/\rho}], \qquad \operatorname{Re} \omega > 0 \quad (2.17)$$

so $\epsilon_k(\omega)$ is a measure of the error of the process. Further let

$$e^{Q(\omega)}\eta(\omega) = -a(\omega) - \Delta[e^{Q(\omega)}s_k^*(\omega)], \qquad \operatorname{Re} \omega > 0. \quad (2.18)$$

Thus, once $s_k^*(\omega)$ has been calculated, $\eta(\omega)$ is known. As $n \to \infty$ we have

$$e^{Q(\omega)}\eta(\omega) \sim \Delta[e^{Q(\omega)}(s^*(\omega) - s_k^*(\omega))] = e^{Q(\omega)}O(\omega^{\theta^*-k/\rho}) \qquad (2.19)$$

at least.

Hence for $\operatorname{Re}(\omega) \geqslant N > 0$ we can determine η_N such that

$$|\eta(\omega)| \leqslant \eta_N |\omega|^{\operatorname{Re}\theta^*-k/\rho}, \qquad \operatorname{Re}(\omega) \geqslant N > 0. \qquad (2.20)$$

It is easily verified that $\epsilon_k(\omega)$ satisfies

$$q(\omega+1)\,\epsilon_k(\omega+1) - q(\omega)\,\epsilon_k(\omega) = e^{Q(\omega)}\eta(\omega), \qquad \operatorname{Re}\omega > 0, \quad (2.21)$$

where

$$q(\omega) = e^{Q(\omega)}\omega^{-k/\rho}, \qquad \operatorname{Re}\omega > 0. \qquad (2.22)$$

Thus,

$$\epsilon_k(\omega) = -\omega^{k/\rho}\,e^{-Q(\omega)}\left\{\sum_{s=0}^{\infty} e^{Q(\omega+s)}\,\eta(\omega+s) + M\right\}, \qquad \operatorname{Re}\omega > 0, \quad (2.23)$$

as may be verified by differencing. Furthermore, M must be zero, since $\epsilon_k(\omega) \to 0$ as $n \to \infty$ by Theorem 1. Then

$$|\epsilon_k(\omega)| \leqslant |\omega|^{k/\rho}\,e^{-\operatorname{Re}Q(\omega)}\,\eta_N \sum_{s=0}^{\infty} e^{\operatorname{Re}Q(\omega+s)}\,|\omega+s|^{\operatorname{Re}Q^*-k/\rho}, \quad \operatorname{Re}\omega \geqslant N. \qquad (2.24)$$

We distinguish two cases. First, let $\operatorname{Re} Q = 0$. Then $\theta^* = \theta + 1$ and

$$|\epsilon_k(\omega)| \leqslant |\omega|^{k/\rho}\,\eta_N \sum_{s=0}^{\infty} |\omega+s|^{\operatorname{Re}\theta+1-k/\rho}. \qquad (2.25)$$

Next, let $\operatorname{Re} Q \not\equiv 0$. Then there exists an $N' \geqslant N$ such that

$$e^{\operatorname{Re}Q(\omega)}\,|\omega|^{\operatorname{Re}\theta-(k/\rho)+2} \qquad (2.26)$$

is monotone decreasing for $\omega \geqslant N'$. Thus

$$|\epsilon_k(\omega)| \leqslant \eta_N |\omega|^{\operatorname{Re}\theta^*+2} \sum_{s=0}^{\infty} |\omega+s|^{-2}, \quad \omega > N'. \qquad (2.27)$$

Our next theorem follows from the expressions (2.25) and (2.27) by an application of the inequality [11, p. 33].

$$\sum_{s=0}^{\infty} \frac{1}{|x+s|^{k'}} < \frac{\pi}{2|x-1|^{k'-1}}, \quad \operatorname{Re} x > 1, \quad k' > 1. \qquad (2.28)$$

THEOREM 3. *Let η_N, $\eta(\omega)$, $\epsilon_k(\omega)$, etc. be as above.*

(A) *Let* Re $Q(\omega) \equiv 0$, $k \geqslant \rho$, Re $\omega \geqslant N > 1$. *Then*

$$|\,\epsilon_k(\omega)| \leqslant (\pi/2)\,\eta_N\,|\,\omega\,|^{k/\rho}\,|\,\omega - 1\,|^{\mathrm{Re}\,\theta+2-k/\rho}. \tag{2.29}$$

(B) *Let* Re $Q(\omega) \not\equiv 0$, *N' be as above. Then*

$$|\,\epsilon_k(\omega)| \leqslant (\pi/2)\,\eta_N\,|\,\omega\,|^{\mathrm{Re}\,\theta^*+2}\,|\,\omega - 1\,|^{-1},\ \mathrm{Re}\,\omega \geqslant N' \geqslant N > 1, \tag{2.30}$$

where θ^* is given by (2.7).

Remark. In either case, we can say that $\epsilon_k(\omega) = O[\omega^{\theta^*+1}]$ as $n \to \infty$. Also, of interest is the case where $\sum a_n$ diverges. The following theorem gives an asymptotic expansion for the nth partial sum of the series.

THEOREM 4. *Let $\sum a_n$ diverge. Let*

$$S_n = S(\omega) \tag{2.31}$$

and a_n, $a(\omega)$, $s^*(\omega)$ be as in Theorem 1. *Then*

$$S(\omega) \sim -e^{Q(\omega)}s^*(\omega) + C, \qquad n \to \infty, \tag{2.32}$$

unless $Q \equiv 0$ and $S(\omega)$ contains a term ω^{-1}. In this case,

$$S(\omega) \sim M \ln \omega + \hat{s}(\omega) + C, \qquad n \to \infty, \tag{2.33}$$

where $\hat{s}(\omega)$ is a series of the kind (2.6) *with $\theta^* = \theta + 1$.*

Proof. Use the equation

$$\Delta S(\omega) = a(\omega), \tag{2.34}$$

and proceed as before. The statement for the case $Q \equiv 0$ follows from an observation of Birkhoff [10, p. 220].

When $Q = \mu_1\omega$, the coefficients β_m in Theorem 1 can be found in closed form as follows. In forming $\Delta R(\omega)$ we encounter the sum

$$\sum_{m=0}^{\infty} \beta_m \omega^{-m/\rho}\left(1 + \frac{1}{\omega}\right)^{\theta^*-m/\rho}$$
$$= \sum_{m=0}^{\infty} \beta_m \omega^{-m/\rho} \sum_{r=0}^{\infty} \frac{\left(\dfrac{m}{\rho} - \theta^*\right)_{r/\rho} (-1)^{r/\rho}\, e_r \omega^{-r/\rho}}{\Gamma((r/\rho) + 1)} \tag{2.35}$$

where

$$e_r = \begin{cases} 1, & \rho \mid r \\ 0, & \text{otherwise.} \end{cases} \tag{2.36}$$

It is found that the β_j's must satisfy the equation

$$e^{\mu_1} \sum_{m=0}^{r} \frac{\Gamma\left(\frac{r}{\rho} - \theta^*\right)(-1)^{(r-m)/\rho}\, e_{r-m}\beta_m}{\Gamma\left(\frac{m}{\rho} - \theta^*\right)\Gamma\left(\frac{r-m}{\rho} + 1\right)} - \beta_r = \begin{cases} -\alpha_r\,, \mu_1 \neq 0 \\ -\alpha_{r-\rho}\,, \mu_1 = 0 \end{cases} \tag{2.37}$$

This equation may be solved by generating functions. First let

$$\begin{gathered} Z(\omega) = \sum_{r=0}^{\infty} \omega^{r/\rho}\zeta_r\,, \qquad A(\omega) = \sum_{r=0}^{\infty} \omega^{r/\rho}A_r\,, \\ \zeta_r = \beta_r/\Gamma((r/\rho) - \theta), \qquad A_r = -\alpha_r/\Gamma((r/\rho) - \theta). \end{gathered} \tag{2.38}$$

Multiplying both sides by $\omega^{r/\rho}$ and summing from $r = 0$ to $r = \infty$ gives the formal relationship

$$e^{\mu_1}Z(\omega) \sum_{r=0}^{\infty} \frac{\omega^{r/\rho}(-1)^{r/\rho}\, e_r}{\Gamma((r/\rho) + 1)} - Z(\omega) = A(\omega) \tag{2.39}$$

or

$$\begin{aligned} Z(\omega) &= A(\omega)(e^{\mu_1 - \omega} - 1)^{-1} = A(\omega) \sum_{s=0}^{\infty} \frac{e^{\omega}(e^{\omega} - 1)^s}{(e^{\mu_1} - 1)^{s+1}} \\ &= A(\omega) \sum_{\nu=0}^{\infty} \omega^{\nu/\rho} e_\nu \sum_{s=0}^{\nu/\rho} \frac{B^{(-s)}_{(\nu/\rho)-s}(1)}{((\nu/\rho) - s)!\,(e^{\mu_1} - 1)^{s+1}} \end{aligned} \tag{2.40}$$

By [12, p. 145] we have

$$B^{(-s)}_{(\nu/\rho)-s}(1) = \frac{(1 + \nu/\rho)\, B_{(\nu/\rho)-s}}{(s+1)}, \qquad \rho \mid \nu, \tag{2.41}$$

so selecting the coefficient of $\omega^{r/\rho}$ on the right-hand side gives

$$\begin{aligned} \beta_r &= -\sum_{\nu=0}^{r} \alpha_{r-\nu} \left(\frac{r-\nu}{\rho} - \theta\right)_{\nu/\rho} e_\nu (1 + \nu/\rho) \sum_{s=0}^{\nu/\rho} \\ &\quad \times \frac{B^{(-s-1)}_{(\nu/\rho)-s}}{(s+1)((\nu/\rho) - s)!\,(e^{\mu_1} - 1)^{s+1}} \\ &= -\sum_{m=0}^{\leqslant r/\rho} \alpha_{r-m\rho} \left(\frac{r}{\rho} - m - \theta\right)_m (m+1) \sum_{s=0}^{m} \frac{B^{(-s-1)}_{m-s}}{(s+1)\,(m-s)!\,(e^{\mu_1} - 1)^{s+1}} \end{aligned} \tag{2.42}$$

here $(\alpha)_\sigma$ is Pochhammer's symbol,

$$(\alpha)_\sigma = \Gamma(\alpha + \sigma)/\Gamma(\alpha). \tag{2.43}$$

For $\mu_1 = 0$, a similar analysis gives

$$\beta_r = \Gamma((r/\rho) - \theta - 1) \sum_{m=0}^{\leqslant r/\rho} \frac{\alpha_{r-m\rho} B_m}{m!\, \Gamma((r/\rho) - m - \theta)}. \tag{2.42}$$

III. Applications to Expansions in Orthogonal Polynomials

The same method can be used to sum series of the form

$$f(x) = \sum_{n=0}^{\infty} a_n P_n^{(\alpha,\beta)}(x) \tag{3.1}$$

hen a_n can be represented by a Birkhoff series. This is because $P_n^{(\alpha,\beta)}(x)$ an be represented as a linear combination of two Birkhoff series,

$$P_n^{(\alpha,\beta)}(x) \sim n^{-1/2} \operatorname{Re} \left\{ Ce^{in\phi} \left[1 + \frac{\alpha_1}{n} + \frac{\alpha_2}{n^2} + \cdots \right] \right\} = \frac{V(n) + \overline{V(n)}}{2}, \tag{3.2}$$

$x = \cos \phi, -\pi < \phi < \pi, \qquad n \to \infty.$

The coefficients α_j can be calculated by applying the method of undeter-ined coefficients (described in the introduction) to the recurrence formula or $P_n^{(\alpha,\beta)}(x)$. The connecting constant C and $\bar{C}$, on the other hand, cannot e found this way, but can be read off the known asymptotic formula for $_n^{(\alpha,\beta)}(x)$ [3, vol. 2, p. 198].

To terms of order n^{-2}, we have

$$C = \frac{e^{\gamma\phi i/2 - (\alpha/2 + 1/4)\pi i}}{\pi^{1/2} (\sin \phi/2)^{\alpha+2} (\cos \phi/2)^{\beta+2}}, \tag{3.3}$$

$$_1 = \frac{\left[\left(\frac{\gamma^2}{2} + \frac{1}{4} - \frac{\gamma}{2} \right) \cos \phi + \frac{(\alpha - \beta)(\gamma - 1)}{2} \right] - \left(\frac{\gamma}{2} + \alpha\beta \right) e^{-i\phi}}{-2i \sin \phi}, \tag{3.4}$$

$\gamma = \alpha + \beta + 1.$

We now write

$$a_n \sim e^{Q(\omega)} s(\omega), \tag{3.5}$$

and

$$\Delta R_n^{(1)} \sim V_1(n)\, e^{Q(\omega)} s(\omega), \qquad \omega \to \infty, \qquad (3.6$$

$$\Delta R_n^{(2)} \sim V_2(n)\, e^{Q(\omega)} s(\omega), \qquad \omega \to \infty. \qquad (3.7$$

Now V_1, V_2 can be converted into Birkhoff expansions in $\omega^{1/\rho}$ by lettir $n = \omega - \xi$. The above equations then can be used to determine Birkho expansions for $R_n^{(1)}$ and $R_n^{(2)}$ since the product of two Birkhoff expansior of the above type is a Birkhoff expansion. Finally, we put

$$R_n = R_n^{(1)} + R_n^{(2)} \qquad (3.8$$

All the essential features of the procedure are displayed in the special ca where $\alpha = \beta = -1/2$ (which corresponds to an expansion in Chebyshe polynomials) and $\rho = 1$, $Q(\omega) = \mu_1 \omega$.

We have

$$T_n(x) = (e^{in\phi} + e^{-in\phi})/2, \qquad x = \cos\phi, \qquad (3.9$$

i.e., the expansion (3.2) consists of a single term. Thus

$$\Delta R_n^{(1)} \sim \frac{-e^{i(\omega-\xi)\phi + \mu_1\omega}}{2}\, \omega^\theta \sum_{r=0}^{\infty} \alpha_r \omega^{-r}. \qquad (3.10$$

So

$$R_n^{(1)} \sim -e^{i\omega\phi + \mu_1\omega} \omega^\theta \sum_{r=0}^{\infty} \beta_r^{(1)} \omega^{-r} \qquad (3.11$$

where the $\beta_r^{(1)}$ satisfy the recursion relation (2.37) (with μ_1 replaced b $\mu_1 + i\phi$).

We have

$$\beta_0^{(1)} = \frac{-\alpha_0 e^{-i\xi\phi}}{2(e^{\mu_1 + i\phi} - 1)}\, \beta_1^{(1)} = \frac{e^{-i\xi\phi}}{2}\left[\frac{-\alpha_1}{(e^{\mu_1+i\phi} - 1)} + \frac{\alpha_0 \theta e^{\mu_1 + i\phi}}{(e^{\mu_1+i\phi} - 1)^2}\right], \text{etc.} \quad (3.12$$

The coefficients in $R_n^{(2)}$ are found by replacing ϕ by $-\phi$ above.

$$R_n \sim \omega^\theta e^{\mu_1\omega}[\alpha_0 v_{n,1} + (1/\omega)(\alpha_1 v_{n,1} + \theta\alpha_0 v_{n,2}) + \cdots], \quad n \to \infty, \quad \omega = n +$$

$$v_{n,1} = \frac{T_n(x) - e^{\mu_1} T_{n-1}(x)}{X}, \qquad (3.13$$

$$v_{n,2} = \frac{e^{\mu_1}(T_{n+1}(x) - 2e^{\mu_1} T_n(x) + e^{2\mu_1} T_{n-1}(x))}{X^2},$$

$$X = e^{2\mu_1} - 2x e^{\mu_1} + 1, \quad X \neq 0.$$

s an illustration of this, consider the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, \sigma^n T_n(x)}{n} = \operatorname{Re} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\, (\sigma e^{i\phi})^n}{n}$$

$$= \operatorname{Re} \ln(1 + \sigma e^{i\phi}) = \ln(1 + \sigma^2 + 2\sigma x)^{1/2}, \qquad |\, \sigma \,| < 1. \quad (3.14)$$

ere

$$\xi = 0,\ e^{\mu_1} = -\sigma,\ \theta = -1,\ \alpha_0 = 1,\ \alpha_j = 0,\ j > 0. \qquad (3.15)$$

$$\ln(1 + \sigma^2 + 2\sigma x)^{1/2} = \sum_{r=1}^{n-1} \frac{(-1)^{r+1}\, \sigma^r T_r(x)}{r} + R_n \qquad (3.16)$$

nd

$$R_n \sim \frac{(-\sigma)^n}{n} \left\{ \frac{(T_n(x) + \sigma T_{n-1}(x))}{X} - \frac{\sigma}{n}\, \frac{(T_{n+1}(x) + 2\sigma T_n(x) + \sigma^2 T_{n-1}(x))}{X^2} \right.$$

$$\left. + \cdots \right\}, \qquad (3.17)$$

$$X = \sigma^2 + 2\sigma x + 1, \qquad n \to \infty.$$

nother interesting example is furnished by the series [3, vol. 2, p. 100].

$$J_\nu(2\sigma \,|\, x \,|) = \sum_{n=0}^{\infty} \epsilon_n J_{(\nu/2)-n}(\sigma)\, J_{(\nu/2)+n}(\sigma)\, T_{2n}(x), \quad \sigma > 0, \qquad (3.18)$$

$$\epsilon_n = \begin{cases} 1, & n = 0, \\ 2, & n > 0. \end{cases}$$

$\nu/2$ is integral, the series terminates. Otherwise, convergence is slow. Since

$$_{(\nu/2)+n}(\sigma)\, J_{(\nu/2)-n}(\sigma) = \frac{(\sigma/2)^\nu\, \Gamma(n - \nu/2)(-1)^{n+1} \sin(\pi\nu/2)}{\Gamma((\nu/2) + n + 1)}$$

$$\times\, {}_2F_3 \left(\begin{matrix} 1/2 + \nu/1, & 1 + \nu/2 & \\ 1 + n + (\nu/2), & 1 - n + (\nu/2), & 1 + \nu \end{matrix} \,\middle|\, -\sigma^2 \right)$$

$$(3.19)$$

$_n$ possesses a Birkhoff series representation in even powers of $1/n$ with $_1 = \pi i$, $\theta = -\nu - 1$. The coefficients in the expansion can best be deterined by using the work of Fields to find the asymptotic expansion of the tio of gamma functions above and then expanding the individual terms f the ${}_2F_3$ in powers of $1/n^2$.

$$_n J_{(\nu/2)+n}(\sigma)\, J_{(\nu/2)-n}(\sigma) \sim \frac{2(-1)^{n+1} \sin(\nu\pi/2)(\sigma/2)^\nu\, n^{-\nu-1}}{\pi}$$

$$\times \left[1 + \frac{1}{n^2} \left(\frac{\nu(\nu+1)(\nu+2)}{24} + \frac{\sigma^2(\nu+2)}{4} \right) + \cdots \right], \quad n \to \infty. \quad (3.20)$$

Then

$$J_\nu(2\sigma \mid x \mid) = \sum_{r=1}^{n-1} \epsilon_r J_{(\nu/2)-r}(\sigma)\, J_{(\nu/2)+r}(\sigma)\, T_{2n}(x) + R_n\,, \qquad (3.21$$

$$R_n \sim \frac{2(-1)^{n+1}\sin(\nu\pi/2)(\sigma/2)^\nu n^{-\nu-1}}{\pi}\left\{\frac{(T_{2n}(x) + T_{2n-1}(x))}{2(1+x)}\right.$$
$$\left. - \frac{((\nu+1)\, T_{2n}(x) + 2T_{2n-1}(x) + T_{2n-2}(x))}{4(1+x)^2\, n} + \cdots\right\}, \quad n \to \infty, \quad x \neq -$$
(3.22

This method of summing series of polynomials whose coefficients hav Birkhoff series representations will work whenever the polynomials satisf a difference equation with coefficients rational in n, since such polynomia themselves always have Birkhoff expansions. In particular, all the polync mials of hypergeometric type discussed by Fields and Luke [13, vol. 1, 7. and by Wimp [14] satisfy such difference equations. The method is partic larly useful for summing series of Laguerre polynomials, since these expar sions tend to converge rather slowly. Let $L_n^{(\alpha)}(x)$ be the Laguerre polynomi of degree n. Following the work of Fields and Luke [13, vol. 1, p. 264] w write

$$L_n^{(\alpha)}(x) \sim \operatorname{Re}\{A(x)\, e^{2i(nx)^{1/2}}\, n^{(\alpha/2)-1/4}\, [1 + C_1 n^{-1/2} + C_2 n^{-1} + C_3 n^{-3/2}]\}$$

$$A(x) = \frac{x^{-(\alpha/2)-1/4}}{\pi^{1/2}}\, e^{x/2} e^{-\pi i((\alpha/2)+1/4)}, \quad C_1 = i\psi_1 x^{-1/2}$$

$$C_2 = \frac{\psi_2}{x} - \frac{\psi_1{}^2}{2x} + \frac{\alpha(\alpha+1)}{2}, \qquad (3.2$$

$$\psi_1 = \frac{x^2}{12} - \frac{(\alpha+1)\, x}{2} - \frac{\alpha^2}{4} + \frac{1}{16},$$

$$\psi_2 = \frac{x^2}{16} - (\alpha+1)(2\alpha+1)\,\frac{x}{8} + \frac{\alpha^2}{16} - \frac{1}{64}, \quad x > 0, \quad \alpha > -1, \quad n \to \alpha$$

As a final example, consider the slowly convergent expansion for Tricomi Ψ function [3, vol. II, p. 215]

$$\Gamma(a)\, \Psi(a, \alpha+1; x) = \sum_{n=0}^{\infty} \frac{L_n^{(\alpha)}(x)}{(n+a)}, \qquad a > 0, \qquad -1 < \alpha < 1/2. \quad (3.2$$

Here we let

$$V_n \sim n^\theta e^{2i(nx)^{1/2}}\, [\beta_0 + \beta_1 n^{-1/2} + \beta_2 n^{-1} + \cdots], \qquad (3.2$$

$$\Delta V_n \sim -n^{(\alpha/2)-5/4} A(x)\, e^{i(nx)^{1/2}}$$
$$\times [1 + C_1 n^{-1/2} + (C_2 - a)\, n^{-1} + (C_3 - C_1 a)\, n^{-3/2} + \cdots], \tag{3.26}$$

$$R_n = \operatorname{Re} V_n . \tag{3.27}$$

roceeding as previously, we find that

$$\Gamma(a)\, \Psi(a, \alpha + 1; x) = \sum_{k=0}^{n-1} \frac{L_n^{(\alpha)}(x)}{(n + a)} + R_n , \tag{3.28}$$

$$_n \sim \frac{x^{-(\alpha/2)-3/4} e^{x/2} n^{(\alpha/2)-3/4}}{\sqrt{\pi}} \Big\{ \sin K_n(x) + (nx)^{-1/2} \Big[\Big(\frac{\alpha}{2} - \frac{3}{4} - \frac{x}{2}\Big) \cos K_n(x)$$
$$+ C_1 x^{1/2} \sin K_n(x) \Big] + (nx)^{-1} \Big[\Big(\frac{\alpha}{2} - \frac{5}{4}\Big) x^{1/2} - \frac{C_1}{2} x^{3/2} \Big) \cos K_n(x)$$
$$- \Big(\frac{x^2}{12} - (C_2 - a)\, x + \Big(\frac{\alpha}{2} - \frac{3}{4}\Big)\Big(\frac{\alpha}{2} - \frac{5}{4}\Big)\Big) \sin K_n(x) \Big] + \cdots \Big\} ,$$
$$n \to \infty, \tag{3.29}$$

$$K_n(x) = 2(nx)^{1/2} - \pi \left(\frac{\alpha}{2} + \frac{1}{4}\right), \; C_1 , \, C_2 \text{ as in (3.23).} \tag{3.30}$$

/e could have taken $\omega = n + a$ rather than $\omega = n$, of course.

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JET WIMP and YUDELL L. LUKE

An algorithm for generating sequences defined by nonhomogeneous difference equations

Estratto

DIREZIONE E REDAZIONE:

VIA ARCHIRAFI, 34 - PALERMO (ITALIA)

AN ALGORITHM FOR GENERATING SEQUENCES DEFINED BY NONHOMOGENEOUS DIFFERENCE EQUATIONS (*)

by **Jet Wimp** (Philadelphia, U. S. A.) and **Yudell L. Luke** (Kansas City, U. S. A.)

1. Introduction

In 1952 J. C. P. Miller introduced an algorithm for computing the modified Bessel function $I_n(x)$ [1]. The method was based on an ingenious use of the well-known difference equation

$$(1.1)\qquad M_n(y) \equiv y(n) - \frac{2(n+1)}{x}\, y(n+1) - y(n+2) = 0, \qquad x > 0,\ n \geqslant 0,$$

satisfied by $I_n(x)$ and $(-)^n K_n(x)$. To illustrate the method, take an integer $N \geqslant 0$ and put

$$(1.2)\qquad \Lambda^{(N)}(N+1) = 0; \qquad \Lambda^{(N)}(N) = 1; \qquad M_n(\Lambda^{(N)}) = 0; \qquad 0 \leqslant n \leqslant N-1.$$

Now define

$$(1.3)\qquad \Omega(N) = \sum_{k=0}^{[N/2]} (-)^k \varepsilon_k \Lambda^{(N)}(2k), \qquad \varepsilon_k = \begin{cases} 1, & k = 0; \\ 2, & k > 0; \end{cases}$$

Miller showed that

$$(1.4)\qquad \lim_{N\to\infty} \Lambda^{(N)}(n)/\Omega(N) = I_n(x), \qquad n \geqslant 0,\ x > 0.$$

(*) This work was sponsored by the Atomic Energy Commission under Contract No. AT (11-1) 1619.

Since no values of $I_n(x)$ are required in the computations, the algorithm is a very attractive one, and Miller's work created an enormous amount of interest. A number of papers soon appeared which discussed the extension of Miller's work to the calculation of special functions defined by other difference equations.

By now, a great deal is known about the application of the method to the σ^{th} order linear homogeneous difference equation

$$\sum_{\nu=0}^{\sigma} A_\nu(n)\, y(n+\nu) = 0, \tag{1.5}$$

see [2] and the references given there. For example, if the coefficients $A_\nu(n)$ are polynomials in n (or merely possess Poincaré type asymptotic expansions in n) then the Miller algorithm will converge provided there is a solution $y(n)$ of (1.5) which satisfies

$$\lim_{n\to\infty} n^k y(n)/y^*(n) = 0, \tag{1.6}$$

for all k and any other solution $y^*(n)$ of (1.5) not a constant multiple of $y(n)$. The algorithm then converges to $y(n)$. Roughly speaking, we can say the Miller algorithm converges to the "smallest" solution of (1.5). For instance, asymptotic estimates for the two linearly independent solutions of (1.1) are

$$I_n(x) = \frac{(x/2)^n}{n!}[1+0(n^{-1})], \qquad (-)^n K_n(x) = \frac{(-2/x)^n}{2}\Gamma(n)[1+0(n^{-1})], \tag{1.7}$$

and the Miller algorithm converges to the smaller solution, $I_n(x)$.

Now a rather different situation occurs with the nonhomogeneous difference equation

$$P_n(y) \equiv \sum_{\nu=0}^{\sigma} B_\nu(n)\, y(n+\nu) = h(n). \tag{1.8}$$

First, notice that we may write the above as a $(\sigma+1)^{th}$ order *homogeneous* equation

$$\frac{P_{n+1}}{h(n+1)} - \frac{P_n}{h(n)} = 0, \tag{1.9}$$

and furthermore that the σ linearly independent solutions of $P_n(y) = 0$ and any constant multiple of a particular solution of (1.8) are all solutions of (1.9) owever, there may be a gain in flexibility by computing with the nonhomoneous equation (1.8) rather than (1.9), since *the solution so computed need t be the smallest solution of the related equation,* (1.9). In fact we define in is paper, for a particular nonhomogeneous equation, a form of the Miller gorithm which displays this behavior. The difference equation in question is at satisfied by the coefficients $E(n)$ which occur in the Chebyshev polynomial pansion of the function

$$\text{10)}\quad \left\{\begin{aligned} F(x) &= 2^{\nu}(\mu+\nu+1)\Gamma(\nu+1)x^{-(\mu+\nu+1)}e^{(a-1)x}\int_0^x e^{-at}t^{\mu}I_{\nu}(t)\,dt \\ &= \sum_{n=0}^{\infty} E(n)\,T_n^*(x/z), \\ &\quad z\neq 0,\quad \operatorname{Re}(\mu+\nu)>-1,\quad \nu\neq -1,-2,-3,\ldots. \end{aligned}\right.$$

ere $T_n^*(w)=T_n(2w-1)$ is the shifted Chebyshev polynomial, z is a range rameter and the notation for all other special functions used in this paper that of [3] ([1]).

In Section 2, we show that $E(n)$ satisfies a third order nonhomogeneous fference equation, the nonhomogeneous term of which depends on the coeffients $C(n)$ in the Chebyshev expansion of I_{ν},

$$\text{11)}\qquad I_{\nu}(x)=\frac{(x/2)^{\nu}e^{x}}{\Gamma(\nu+1)}\sum_{n=0}^{\infty}C(n)\,T_n^*(x/z).$$

e also derive auxiliary relationships involving sums of the $E(n)$ and give the cursion formula for $C(n)$.

In Sections 3 and 4, we treat the calculation of $E(n)$ from its difference uation. Section 3 is devoted to the above-mentioned modification of the Miller gorithm, its derivation and convergence properties. The methods of Section 4 e based on the solution of simultaneous linear equations. In the Appendix, e present tables for the coefficients $C(n)$ for $\nu=0$ and $\nu=1$ and $E(n)$ for $=\nu=0$ and $a=0(0.1)1.0$.

Despite the special nature of the problem considered, the ideas behind the ethods given in this paper are quite general, and are capable of application a wide class of difference equations.

([1]) Both x and z may be complex as long as the proper branch of $x^{\mu+\nu+1}$ is taken cause aside from this factor the right-hand side in (1.10) is an entire function of x and z. us the series converges for all x and z, $z\neq 0$. (The restriction $\nu\neq -1,-2,\ldots$, is ually extrinsic to our analysis, since $I_{-m}(t)=I_m(t)$, $m=0,1,2,\ldots$).

2. THE RECURSION FORMULA

Theorem 1. Let $z \neq 0$, $a \neq 1$, $\mathrm{Re}(\mu + \nu) > -1$. Then $E(n)$ satisfies

$$(2.1) \qquad \left\{ \begin{aligned} P_n(E) \equiv {} & \frac{2E(n)}{\varepsilon_n} + \frac{1}{y}(n + \mu + \nu + 2 + y)E(n+1) + \\ & + \frac{1}{y}(n - \mu - \nu + 1 - y)E(n+2) - E(n+3) \\ = {} & \frac{(\mu + \nu + 1)}{y}[C(n+1) - C(n+2)] = h(n), \quad n = 0, 1, 2, .. \end{aligned} \right.$$

where

$$(2.2) \qquad y = \frac{z(1-a)}{4}, \qquad \varepsilon_n = \begin{cases} 1, & n = \\ 2, & n > \end{cases}$$

and $C(n)$ is as in (1.11).

Proof. Differentiate (1.10) with respect to x and use (1.11). Then

$$x F'(x) + \{(\mu + \nu + 1) + 4y(x/z)\} F(x) = (\mu + \nu + 1) \sum_{n=0}^{\infty} C(n) T_n^*(x/z),$$

$$(2.3) \qquad F(x) = \sum_{n=0}^{\infty} E(n) T_n^*(x/z).$$

If we now make liberal use of the relations

$$\sum_{n=0}^{\infty} b_n \frac{d T_n^*(w)}{dw} = \sum_{n=0}^{\infty} d_n T_n^*(w),$$

$$(2.4) \qquad d_n = 2\varepsilon_n \sum_{r=0}^{\infty} (n + 2r + 1) b_{n+2r+1},$$

$$w \sum_{n=0}^{\infty} b_n T_n^*(w) = \sum_{n=0}^{\infty} f_n T_n^*(w),$$

$$(2.5) \qquad f_n = \begin{cases} \frac{1}{4}(2b_0 + b_1), & n = 0 \\ \frac{1}{4}(2b_0 + 2b_1 + b_2), & n = \\ \frac{1}{4}(b_{n-1} + 2b_n + b_{n+1}), & n > \end{cases}$$

n (2.3), the completion of the proof is merely a matter of algebra. Equating oefficients of $T_n^*(x/z)$ in the resulting equation gives

$$(2.6)\qquad (\mu+\nu+1)E(0)+y(2E(0)+E(1))+\frac{1}{4}(2F(0)+F(1))=(\mu+\nu+1)C(0),$$

$$(2.7)\qquad \begin{cases} (\mu+\nu+1)E(n+1)+y\left[\dfrac{2E(n)}{\varepsilon_n}+2E(n+1)+E(n+2)\right]+ \\ +\dfrac{1}{4}\left[\dfrac{2F(n)}{\varepsilon_n}+2F(n+1)+F(n+2)\right]=(\mu+\nu+1)C(n+1), \quad n\geqslant 0, \end{cases}$$

here

$$(2.8)\qquad F(n)=2\varepsilon_n\sum_{r=0}^{\infty}(n+2r+1)E(n+2r+1).$$

ifferencing (2.7) with respect to n gives (2.1). (Equation (2.6) can serve as a heck for computational purposes). An alternative approach is to convert (2.3) to the integral equation

$$(2.9)\qquad \begin{cases} xF(x)+(\mu+\nu+1)\displaystyle\int_0^x F(t)\,dt+(a+1)\int_0^x tF(t)\,dt= \\ =(\mu+\nu+1)\displaystyle\sum_{n=0}^{\infty}C(n)\int_0^x T_n^*(t/z)\,dt. \end{cases}$$

ow employ the relations

$$(2.10)\qquad \begin{cases} \displaystyle\sum_{n=0}^{\infty}g_n\int_0^x T_n^*(w)\,dw=\left\{\frac{g_0}{2}-\frac{g_1}{8}-\frac{1}{2}\sum_{r=2}^{\infty}\frac{(-)^r g_r}{r^2-1}\right\}T_0^*(x)+ \\ +\dfrac{1}{4}(2g_0-g_2)T_1^*(x)+\dfrac{1}{4}\displaystyle\sum_{n=2}^{\infty}\frac{(g_{n-1}-g_{n+1})}{n}T_n^*(x), \end{cases}$$

nd (2.4) in (2.9) and equate coefficients of $T_n^*(x/z)$. We get the equations

$$(2.11)\qquad \begin{cases} \dfrac{1}{4}\{2E(0)+E(1)\}+(\mu+\nu)\left\{\dfrac{E(0)}{2}-\dfrac{E(1)}{8}-\dfrac{1}{2}\displaystyle\sum_{n=2}^{\infty}\frac{(-)^n E(n)}{n^2-1}\right\}+ \\ +y\left\{\dfrac{3E(0)}{4}+\dfrac{E(1)}{12}-\dfrac{19E(2)}{48}+3\displaystyle\sum_{n=3}^{\infty}\frac{(-)^n E(n)}{(n^2-1)(n^2-4)}\right\}= \\ =(\mu+\nu+1)\left\{\dfrac{C(0)}{2}-\dfrac{C(1)}{8}-\dfrac{1}{2}\displaystyle\sum_{n=2}^{\infty}\frac{(-)^n C(n)}{n^2-1}\right\}, \end{cases}$$

$$(2.12)\quad \begin{cases} E(0)(2\mu+2\nu+2+4y)+E(1)(2+y)+E(2)(1-\mu-\nu-2y)-yE(3)= \\ \qquad = (\mu+\nu+1)\{2C(0)-C(2)\}, \end{cases}$$

and an equation which follows by adding (2.1) to itself with n replaced b $n+1$. Equations (2.11) and (2.12) are useful for check purposes and for con puting $E(n)$ by the solution of systems of linear equations (see Section 4).

The coefficients $C(n)$ can be expressed in closed form in terms of hype geometric functions

$$(2.13)\qquad C(n)=\frac{\varepsilon_n\left(\nu+\frac{1}{2}\right)_n(-)^n z^n}{2^n(2\nu+1)_n n!}\,{}_2F_2\left(\begin{matrix} n+\nu+\frac{1}{2},\ n+\frac{1}{2} \\ n+2\nu+1,\ 2n+1\end{matrix}\,\middle|\,-2z\right),$$

by expressing $e^{-x}x^{-\nu}I_\nu(x)$ as a confluent hypergeometric function ${}_1F_1$ and usi the results in [4]. Then from the work of Wimp, [5], we get a theorem for $C($ analogous to that for $E(n)$.

Theorem 2. Let $z\neq 0$, $\nu\neq -\frac{1}{2}, -\frac{3}{2}, \ldots$. Then $C(n)$ satisfies the r cursion relation

$$(2.14)\quad \begin{cases} Q_n(C)\equiv \dfrac{2C(n)}{\varepsilon_n}+\dfrac{(n+1)}{(n+2)\left(n+\nu+\frac{1}{2}\right)}\times \\ \times\left[\left(n-\nu+\frac{5}{2}\right)+\dfrac{2(n+2\nu+1)(n+2)}{z}\right]C(n+1)- \\ -\left[1-\dfrac{2(n+1)(n-2\nu+2)}{\left(n+\nu+\frac{1}{2}\right)z}\right]C(n+2)- \\ -\dfrac{(n+1)\left(n-\nu+\frac{5}{2}\right)}{(n+2)\left(n+\nu+\frac{1}{2}\right)}C(n+3)=0. \end{cases}$$

3. Computational scheme I

We now describe an interesting modification of Miller's algorithm whic under certain conditions can be used to compute $E(n)$ in the backward directio from the nonhomogeneous difference equation (2.1). The idea is as follow One generates first a certain solution of (2.1), and next a certain solution the homogeneous equation $P_n=0$. Then one seeks to determine a linear comb nation of these two solutions which closely approximates $E(n)$.

To be more specific, assume we are given an integer $N \geqslant 0$. We define o sequences $\{e^{(N)}(n)\}$, $\{f^{(N)}(n)\}$ by the formulae

$$\text{1)} \quad \begin{cases} e^{(N)}(N+2)=e^{(N)}(N+1)=0; \;\; e^{(N)}(N)=1; \;\; P_n\{e^{(N)}\}=0, & 0\leqslant n\leqslant N-1; \\ f^{(N)}(N+3)=f^{(N)}(N+2)=f^{(N)}(N+1)=0; \;\; P_n\{f^{(N)}\}=h(n), & 0\leqslant n\leqslant N. \end{cases}$$

w, consider the sequence

$$T^{(N)}(n) = f^{(N)}(n) + V_N e^{(N)}(n), \qquad 0 \leqslant n \leqslant N+2$$

$$\text{2)} \qquad V_N = \frac{\left[1 - \sum_{k=0}^{N} (-)^k f^{(N)}(k)\right]}{\sum_{k=0}^{N} (-)^k e^{(N)}(k)}.$$

Our hope is that $T^{(N)}(n)$ will approach $E(n)$ fairly rapidly as $N \to \infty$. e can prove a theorem to this effect, but we first require two lemmas.

In what follows, $\mathfrak{A}(\gamma_0, n)$ will be a generic notation for the following ymptotic series in n

$$\text{3)} \qquad \mathfrak{A}(\gamma_0, n) = \gamma_0 + \gamma_1 n^{-1} + \gamma_2 n^{-2} + \cdots, \qquad \gamma_0 \neq 0, \; n \to \infty,$$

ere "generic" means that $\mathfrak{A}$ need not involve the same values of $\gamma_0, \gamma_1, \ldots,$ erever the symbol appears.

Lemma 1

$$\text{4)} \qquad {}_2F_2\left(\left.\begin{matrix} n+a_1, \; n+a_2 \\ n+b, \; 2n+\gamma \end{matrix}\right| z\right) = g(n) \sim e^{z/2}\,\mathfrak{A}(1, n).$$

Proof. The lead term, $e^{z/2}$, is computed by an easily justified term-by-term ssage to the limit in the series definition of $g(n)$. To show the existence a complete asymptotic expansion of the desired kind, we note that $g(n)$ isfies a third order difference equation with coefficients which are rational ctions of n, see [5, 6, 7,]. The Birkhoff-Trjitzinsky analytic theory of diffe- ice equations [8], [9], asserts that this equation possesses three linearly lependent solutions which have distinct (apart from a constant multiple) asymptotic representations of the form

$$
(3.5) \qquad \begin{cases} W(n) = n^{\mu n + \theta}\, e^{K(n)}\, \mathfrak{A}(\gamma_0,\, n^{\frac{1}{\varrho}}), \quad \rho \text{ an integer} \geqslant 1, \\ K(n) = \alpha_1 n^{\frac{\varrho-1}{\varrho}} + \alpha_2 n^{\frac{\varrho-2}{\varrho}} + \cdots + \alpha_{\varrho-1} n^{\frac{1}{\varrho}}, \end{cases}
$$

or possibly a linear combination of such series multiplied by logarithmic tern $(\ln n)^h$, $h = 0, 1, 2, \ldots$. The form of the series and the actual coefficients determined by substitution using the techniques of [8, p. 210 ff] [1]. For t present case, the computations are carried out in [5], where it is shown tl $\rho = 1$ in each asymptotic series pertaining to the difference equation for g(that there are three possible values of μ (2, 1 and 0), associated with each is value of λ ($-16/e^2z^2$, $-4/ez$, and 1, respectively) and that no logarith occur [2]. Clearly, the series corresponding to $\mu = 0$ uniquely describes g(

Lemma 2. Let $a \neq \pm 1$, $z \neq 0$, $\operatorname{Re}(\mu + \nu) > -1$. Then for h = 0, 1, 2, linearly independent functions $\varphi_h(n)$ can be determined with the asympto representations

$$
(3.6) \qquad \begin{cases} \varphi_h(n) \sim C_h \lambda_h^n\, n^{(\mu_h \ln n + \theta_h)}\, \mathfrak{A}(1,\, n) \\ \lambda_0 = -ez/2; \quad \mu_0 = -1; \quad \theta_0 = -\nu - 2; \quad C_0 = \dfrac{e^{-z}\, 2^{2\nu + 3/2}\, \Gamma(\nu+1)(\mu+\nu+1}{\pi(a+1)} \\ \lambda_1 = 4/ez(1-a); \quad \mu_1 = 1; \quad \theta_1 = -\mu - \nu - 3/2; \quad C_1 = 1; \\ \lambda_2 = -1; \quad \mu_2 = 0; \quad \theta_2 = 2\mu + 2\nu + 1; \quad C_2 = 1; \\ \lambda_3 = -ez(1-a)/4; \quad \mu_3 = -1; \quad \theta_3 = -\mu - \nu - 3/2; \quad C_3 = e^{-\frac{z(1-a)}{2}} \sqrt{\dfrac{2}{\pi}}\, \Gamma(\mu+\nu+ \end{cases}
$$

---

(1) Essentially, one substitutes the relationship

$$
\frac{W(n+k)}{W(n)} \sim n^{\mu k} \lambda^k \exp\left\{k\left(\mu + \frac{\partial K(n)}{\partial n}\right)\right\}\left\{1 + \frac{k}{2n}(2\theta + \mu) + n^{-1-\frac{1}{\varrho}}\, \mathfrak{A}(\gamma_0,\, n^{1/\varrho})\right\}
$$

into the equation and then formally compares coefficients of like powers of n to determ ρ, α_1, α_2, $\ldots$, $\alpha_{\varrho-1}$, θ, γ_1, γ_2, γ_3, $\ldots$. Alternatively one may, in the above case, determ the γ_i directly by using the asymptotic expansion

$$
\frac{\Gamma(n + \sigma + k)}{\Gamma(n + \sigma)} \sim n^k\, \mathfrak{A}(1,\, n)
$$

in the coefficients of the power series for $g(n)$ qua function of z.

(2) For logarithms to occur, certain conditions must be satisfied, see Wimp, [2]. For instar μ, λ and K for two of the series (3.5) must be the same.

› that φ_1, φ_2, φ_3 satisfy the homogeneous equation $P_n = 0$ and $\varphi_0(n) \equiv E(n)$ ıtisfies (2.1).

Define

.7) $$S_h(N) = \sum_{k=0}^{N} (-)^k \varphi_h(k).$$

hen

.8) $$S_0(N) \sim 1 + (-)^N E(N+1)\mathfrak{A}(1, N),$$

.9) $$S_1(N) \sim (-)^N \varphi_1(N)\mathfrak{A}(1, N),$$

.10) $$S_2(N) \sim d_1 + d_2 \ln N + (-)^N N\varphi_2(N)\mathfrak{A}(\gamma_0, N),$$

.11) $$S_3(N) \sim 1 + (-)^N \varphi_3(N+1)\mathfrak{A}(1, N),$$

here d_1 and d_2 are constants with $d_2 = 0$ if and only if $2\mu + 2\nu + 1 =$ $= 0, 1, 2, \ldots$.

Proof. To show the existence of such solutions, we form from (2.1) the th order linear homogeneous difference equation

.12) $$\frac{h_n P_{n+1}(\Phi)}{h_{n+1}} - P_n(\Phi) = 0.$$

Since, by lemma 1,

.13) $$C(n) \sim \frac{\Gamma(\nu+1)\, 2^{2\nu+1/2}}{\pi} e^{-z}\left(-\frac{ez}{2}\right)^n n^{-n-\nu-1}\mathfrak{A}(1, n),$$

e have

.14) $$h(n)/h(n+1) = \Delta C(n+1)/\Delta C(n+2) \sim n\mathfrak{A}(\gamma_0, n),$$

here Δ is the forward difference operator.

Thus, each coefficient in the equation (3.12) has an asymptotic represen- tion of the form $n^m \mathfrak{A}(\gamma_0, n)$, $m = 0, 1, 2$. The Birkhoff-Trjitzinsky theory en shows that the equation possesses four linearly independent solutions ›rresponding to four distinct asymptotic series of the previously mentioned pe, provided $y \neq 0$. As before, the parameters in the series are determined / substitution into (2.1) or (3.12). The constant C_0 may be determined from .1), while C_1 and C_2 are chosen (arbitrarily) to be 1.

We now turn to the solution $\varphi_3(n)$. That the function

$$(3.15)\quad \begin{cases} \zeta(n) = \left[-\dfrac{z(1-a)}{4}\right]^n \dfrac{\varepsilon_n}{(\mu+\nu+2)_n}\, {}_2F_2\left(\begin{matrix} n+\frac{1}{2},\ n+1 \\ n+\mu+\nu+2,\ 2n+1 \end{matrix}\middle| -z(1-a)\right) \\ \sim \sqrt{\dfrac{2}{\pi}}\, e^{-z(1-a)/2}\,\Gamma(\mu+\nu+2)\left[-\dfrac{ez(1-a)}{4}\right]^n n^{-n-\mu-\nu-3/2}\,\mathfrak{A}(1, n) \end{cases}$$

satisfies the homogeneous difference equation $P_n = 0$ follows from the wor of Wimp [5], and the expansion

$$(3.16)\qquad {}_1F_1\left(\begin{matrix} 1 \\ \mu+\nu+2 \end{matrix}\middle| -wz(1-a)\right) = \sum_{k=0}^{\infty} \zeta(k)\, T_k^*(w)$$

follows from [4]. Setting $w = 0$ gives

$$(3.17)\qquad 1 = \sum_{k=0}^{\infty} (-)^k \zeta(k).$$

Since φ_3 is exponentially subdominant to φ_1 and φ_2, we can choose C_3 to mak the unique identification $\zeta(k) \equiv \varphi_3(k)$.

Next, we use Lemma 1 to get the asymptotic expansion in (3.15) and th estimates (3.8) and (3.9) are easily obtained. To show (3.10) requires extensiv use of the results in [2]. We first note that $S_2(N)$ satisfies the difference equatio

$$(3.18)\qquad \Phi(N+2) + \Phi(N+1)(\tau_N - 1) - \tau_N \Phi(N) = 0, \qquad \tau_N = \frac{\varphi_2(N+2)}{\varphi_2(N+1)}.$$

Any solution of this equation must have an asymptotic representation of the forr

$$(3.19)\qquad C_1 n^{\theta+r}\,\mathfrak{A}(1, n) + (C_1 \ln n + C_2)\, n^{\theta}\,\mathfrak{A}(1, n),$$

where r is an integer, see [2]. Actual substitution shows the series multiplyin the log term must be a constant and in fact is zero unless $2\mu + 2\nu + 1 =$ $= -1, 0, 1, 2, \ldots$. The resulting asymptotic series can be expressed in th form given by (3.10).

For numerical investigations, it is convenient to have the asymptotic esti mates for $C(n)$ and $E(n)$ expressed in the following form:

$$(3.20)\qquad C(n) \sim \frac{2\left(\nu+\frac{1}{2}\right)_n (-)^n (z/2)^n}{n!\,(2\nu+1)_n}\, e^{-z}\left\{1 + \frac{a_1}{n} + \frac{a_2}{n^2} + \cdots\right\},$$

$$E(n) \sim \frac{2\left(\nu+\frac{3}{2}\right)_n (-)^n (\mu+\nu+1)(z/2)^n e^{-z}}{(n+1)!\,(2\nu+2)_n (a+1)} \left\{1+\frac{b_1}{n}+\frac{b_2}{n_2}+\cdots\right\}, \tag{3.21}$$

where

$$\begin{cases} a_1 = \dfrac{z^2}{4}+\left(\nu+\dfrac{1}{2}\right)z, \\ a_2 = \dfrac{z^4}{32}+\left(\nu+\dfrac{1}{2}\right)\dfrac{z^3}{4}+\left(\nu^2+\nu-\dfrac{1}{4}\right)\dfrac{z^2}{2}-\left(\nu+\dfrac{1}{2}\right)\dfrac{z}{2}+\dfrac{(4\nu+1)(2\nu+1)}{8}, \\ b_1 = \dfrac{1-2\mu}{1+a}+\dfrac{z^2}{4}+\left(\nu+\dfrac{3}{2}\right)z, \\ b_2 = \dfrac{z^4}{32}+(2\nu+3)\dfrac{z^3}{8}+(4\nu^2+12\nu+7)\dfrac{z^2}{8}+\dfrac{(1-2\mu)}{(1+a)}\dfrac{z^2}{4}- \\ \qquad -(10\nu+13)\dfrac{z}{4}+\dfrac{(2\nu+5)(1-2\mu)z}{2(1+a)}+\dfrac{(2\nu+1)(4\nu+1)}{8}+ \\ \qquad +\dfrac{(2\mu-3)(2\mu-1)}{(1+a)^2}-\dfrac{(2\nu^2+\nu+2)}{(1+a)}+\dfrac{(2\nu+5)\mu}{(1+a)}. \end{cases} \tag{3.22}$$

We are now in a position to prove

Theorem 3. Let $a \neq 1$, $z \neq 0$, $\mathrm{Re}(\mu+\nu) > -1$, $\nu \neq -1, -2, \ldots$. Then for some θ,

$$T^{(N)}(n) = E(n) + 0\left\{\left(\frac{ez}{2}\right)^N N^{\theta-N}\right\}, \qquad N \to \infty. \tag{3.23}$$

Proof. Since $T^{(N)}(n)$ clearly satisfies (2.1), we may write

$$T^{(N)}(n) = \sigma_1\varphi_1(n) + \sigma_2\varphi_2(n) + \sigma_3\varphi_3(n) + E(n), \tag{3.24}$$

where the σ_h are constants (depending on N but not on n). Letting $n = N+2$, and $N+1$ above give two equations for the determination of σ_1, σ_2, σ_3. A third equation can be obtained by multiplying both sides by $(-)^n$ and summing from $n = 0$ to N. The resulting system is

$$\begin{cases} 0 = \sigma_1\varphi_1(N+2) + \sigma_2\varphi_2(N+2) + \sigma_3\varphi_3(N+2) + E(N+2) \\ 0 = \sigma_1\varphi_1(N+1) + \sigma_2\varphi_2(N+1) + \sigma_3\varphi_3(N+1) + E(N+1) \\ 1 = \sigma_1 S_1(N) + \sigma_2 S_2(N) + \sigma_3 S_3(N) + S_0(N). \end{cases} \tag{3.25}$$

To determine the behavior of σ_h as $N \to \infty$, we solve the above system o equations by Cramer's rule and use the relation

$$\frac{\varphi_h(N+r)}{\varphi_h(N)} \sim (Ne)^{\mu_h r} \lambda_h^r \mathfrak{A}(1, N) \tag{3.26}$$

in the determinants involved. (Because of the linear independence of the φ none of these determinants can vanish for N large enough). For example, th determinant of the system Δ has the asymptotic expansion

$$\Delta \sim -\lambda_1^2 N^2 \varphi_1(N) \varphi_2(N) \mathfrak{A}(1, N). \tag{3.27}$$

Three more straightforward computations give

$$\left\{\begin{aligned} &\sigma_1 \sim \frac{-\lambda_0 E(N)}{\lambda_1^2 N^3 \varphi_1(N)} \mathfrak{A}(1, N), \\ &\sigma_2 \sim \frac{\lambda_0 E(N)}{N \varphi_2(N)} \mathfrak{A}(1, N), \\ &\sigma_3 \sim E(N)\Big((-)^N \mathfrak{A}(\gamma_0, N) + \frac{\lambda_0 (d_1 + d_2 \ln N)}{N \varphi_2(N)} \mathfrak{A}(\gamma_0, N)\Big). \end{aligned}\right. \tag{3.28}$$

Substituting these estimates in (3.24) and using (3.6) give the theorem.

The restriction from Lemma 2 that $a \neq -1$ can be relaxed by a mo detailed analysis which would permit the occurrence of logarithmic terms i the solutions $\varphi_0(n)$, $\varphi_3(n)$. However, we omit details. For similar computation see Wimp [2].

Since the values of $C(n)$ are required for the application of Theorem we now show how the ordinary Miller algorithm may be used to obtain ther For an integer $N \geqslant 0$, define the sequence $\{q^{(N)}(n)\}$ by

$$\left\{\begin{aligned} &q^{(N)}(N+2) = q^{(N)}(N+1) = 0; \quad q^{(N)}(N) = 1; \\ &Q_n(q^{(N)}) = 0, \qquad 0 \leqslant n \leqslant N-1, \end{aligned}\right. \tag{3.29}$$

and consider the sequence

$$U^{(N)}(n) = \frac{q^{(N)}(n)}{\sum_{k=0}^{N} (-)^k q^{(N)}(k)}, \qquad 0 \leqslant n \leqslant N + \tag{3.30}$$

We have

Theorem 4. Let $z \neq 0$, $\nu \neq -\frac{1}{2}, -\frac{3}{2}, \ldots$. Then for some θ

31) $$U^{(N)}(n) = C(n) + 0\left\{\left(\frac{ez}{2}\right)^N N^{\theta - N}\right\}, \qquad N \to \infty.$$

Proof. Actual computation shows that there exists a set of linearly inde-ndent solutions of (2.14) with distinct asymptotic representations of the form

$$\varphi_h(n) \sim n^{(\mu_h n + \theta_h)} \lambda_h^n \mathfrak{A}(\gamma_0, n),$$

32) $$\begin{cases} \mu_1 = 1; \quad \lambda_1 = 2/ez; \quad \theta_1 = -\nu - 1; \\ \mu_2 = 0; \quad \lambda_2 = -1; \quad \theta_2 = 4\nu - 1; \\ \mu_3 = -1; \quad \lambda_3 = -ez/2; \quad \theta_3 = -\nu - 1. \end{cases}$$

early, $\varphi_3(n)$ can be taken to be $C(n)$. Since $q^{(N)}(n)$ satisfies (2.14), we may write

33) $$q^{(N)}(n) = \tau_1 \varphi_1(n) + \tau_2 \varphi_2(n) + \tau_3 \varphi_3(n).$$

e find the three equations needed to determine the τ_h to be

34) $$\begin{cases} 0 = \tau_1 \varphi_1(N+2) + \tau_2 \varphi_2(N+2) + \tau_3 \varphi_3(N+2) \\ 0 = \tau_1 \varphi_1(N+1) + \tau_2 \varphi_2(N+1) + \tau_3 \varphi_3(N+1) \\ 1 = \tau_1 S_1(N) + \tau_2 S_2(N) + \tau_3 S_3(N). \end{cases}$$

The system is solved by Cramer's rule and asymptotic estimates for the are easily obtained. We omit details, but the result of using these estimates (3.30) is the Theorem.

Although the algorithm for calculating $E(n)$ possesses excellent theoretical nvergence properties, the error being of the order of $(z/2)^N/N!$, see (3.23), its plication can give rise to rather serious numerical instabilities, the nature of hich is made clear by the easily derived formulae

35) $$f^{(N)}(n) \sim E(n) - \left(\frac{2}{1-a}\right)^{N+1} N^{\mu - \frac{1}{2}} \mathfrak{A}(1, N) \varphi_3(n),$$

36) $$e^{(N)}(n) \sim \left(-\frac{4N}{ez(1-a)}\right)^N N^{\mu + \nu + 3/2} \mathfrak{A}(1, N) \varphi_3(n),$$

where $\mathfrak{A}(1, N)$ is independent of n. For certain ranges of values of a, $(-1$ $< a < 3)$, both the above quantities grow rapidly as $N \to \infty$, and we are the position of having to determine a linear combination of the two which small. It is clear that there will occur for these values of a an appreciable lo of significant figures, a loss which grows worse as a approaches 1.

To illustrate this phenomenon, let us define as a measure of the loss significance the quantity

$$\omega = \left| 1 - \sum_{k=0}^{N} (-)^k \tilde{T}^{(N)}(k) \right|, \tag{3.37}$$

where $\tilde{T}^{(N)}(k)$ is the actual *computed* $T^{(N)}(k)$. If all calculations are done w infinite precision, ω will be zero. When 16 places are carried in the comput tions and $N = 32$, $z = \infty$, $\mu = \nu = 0$, ω is found to vary with a as in t accompanying table.

Even in the most favorable case $(a = 0)$ it is likely that none of the E computed by the algorithm is accurate to more than 8 places.

All is not lost, however. A tremendous improvement in the efficiency the algorithm can be made by *iterating*, i. e., by using the values $T^{(N)}($ $T^{(N)}(N-1)$, $T^{(N)}(N-2)$ just computed (rather than zeros) as starting valu $f^{(N')}(N'+3)$, $f^{(N')}(N'+2)$, $f^{(N')}(N'+1)$ for a new sequence $f^{(N')}(n)$, $N' = N-$ Then, by taking $e^{(N')}(N'+2) = e^{N'}(N'+1) = 0$, $e^{(N')}(N') = 1$ as before, one c calculate from the appropriate difference equations the new sequences $\{f^{(N')}($ $\{e^{(N')}(n)\}$. This procedure serves to reduce considerably the size of the elemen $f^{(N)}(n)$, and hence to mitigate the loss of significance caused by subtracti large numbers. This procedure can be repeated until no further reduction in occurs. Let ω^* be the smallest value of ω so obtained. The table shows t improvement possible and the number of iterations needed. Two other measur of the accuracy of the final coefficients are the absolute values of the differenc between the left-hand right-hand sides of (2.11) and (2.12). These are tabulat below as ε and δ, respectively. (For the computations we have replaced a infinite sums in (2.11)-(2.12) by $\sum_{k=0}^{N}$ and $E(n)$ where it occurs is approximated $\tilde{T}^{(N)}(n)$). In all the data of Table 3.1, $z = 8$, $\mu = \nu = 0$, and $N = 32$.

Table 3.1

a	ω	ω^*	ε	$\hat{\delta}$
0	$4.05 \cdot 10^{-8}$	$\leqslant 1.00 \cdot 10^{-15}$ (1 iteration)	$2.35 \cdot 10^{-15}$	$9.10 \cdot 10^{-15}$
0.1	$3.92 \cdot 10^{-7}$	$\leqslant 1.00 \cdot 10^{-15}$ (1 iteration)	$1.48 \cdot 10^{-15}$	$6.00 \cdot 10^{-15}$
0.2	$4.16 \cdot 10^{-5}$	$\leqslant 1.00 \cdot 10^{-15}$ (1 iteration)	$2.37 \cdot 10^{-15}$	$9.77 \cdot 10^{-15}$
0.3	$1.74 \cdot 10^{-3}$	$\leqslant 1.00 \cdot 10^{-15}$ (1 iteration)	$2.37 \cdot 10^{-15}$	$9.33 \cdot 10^{-15}$
0.4	$1.45 \cdot 10^{-1}$	$\leqslant 1.00 \cdot 10^{-15}$ (1 iteration)	$2.40 \cdot 10^{-15}$	$8.66 \cdot 10^{-15}$
0.5	$4.90 \cdot 10$	$\leqslant 1.00 \cdot 10^{-15}$ (2 iterations)	$2.33 \cdot 10^{-15}$	$8.44 \cdot 10^{-15}$
0.6	$2.45 \cdot 10^{4}$	$\leqslant 4.00 \cdot 10^{-14}$ (2 iterations)	$3.11 \cdot 10^{-14}$	$1.39 \cdot 10^{-13}$
0.7	$7.54 \cdot 10^{7}$	$\leqslant 1.09 \cdot 10^{-10}$ (2 iterations)	$5.81 \cdot 10^{-11}$	$2.30 \cdot 10^{-10}$
0.8	$7.78 \cdot 10^{14}$	$\leqslant 4.22 \cdot 10^{-5}$ (3 iterations)	$2.72 \cdot 10^{-5}$	$1.17 \cdot 10^{-4}$

The computations were done in double precision on an IBM 360. Normal round-off error limits the accuracy to about 14 decimal places. For $a = 0(0.1)0.5$, the accuracy obtained in the final coefficients (those corresponding to ω^*) is about the best possible for the value $N = 32$. It is clear from the true coefficients given in the Appendix that only 27 coefficients need be recorded since all higher coefficients are less than 10^{-15} in magnitude. Equations (3.35) and (3.36) show that to minimize the adverse effect of instability one should use as small an N as possible consistent with the maximum accuracy that can be achieved from the computer. This suggests the following procedure. For $a = 0.6(0.1)0.8$, take $N = 26$ and with $N' = 23$, use the iteration procedure described above by employing the final values of $T^{(32)}(25)$, $T^{(32)}(24)$ and $T^{(32)}(23)$ obtained in the original iteration procedure for $N = 32$. This technique was followed for $a = 0.6$, 0.7 and 0.8 and the following table describes the results. The notation is the same as that used in Table 3.1.

Table 3.2

a	ω	ω^*	ε	$\hat{\delta}$
0.6	$\leqslant 4.00 \cdot 10^{-14}$	$\leqslant 6.00 \cdot 10^{-16}$ (1 iteration)	$2.18 \cdot 10^{-15}$	$8.94 \cdot 10^{-15}$
0.7	$\leqslant 1.09 \cdot 10^{-10}$	$\leqslant 1.00 \cdot 10^{-15}$ (1 iteration)	$1.65 \cdot 10^{-15}$	$6.77 \cdot 10^{-15}$
0.8	$\leqslant 4.22 \cdot 10^{-5}$	$\leqslant 3.64 \cdot 10^{-13}$ (2 iterations)	$2.69 \cdot 10^{-13}$	$1.16 \cdot 10^{-12}$

Of course, the accuracy attainable diminishes rapidly as $a \to 1.0$. For $a = 0.9$, none of the above approaches is satisfactory, e. g., after 4 iterations, the value of ω^* in Table 3.1 is $9.55 \cdot 10^{3}$. For a near unity, the coefficients were found by the methods of Section 4.

4. COMPUTATIONAL SCHEME II

In Section 3 we developed modifications of Miller's algorithm which per- mitted the calculation of the coefficients $E(n)$ in the backward direction fron the nonhomogeneous difference equation (2.1), and we saw how these method: deteriorated in accuracy as a approached unity.

We now show how the coefficients can be determined for all values of a $a \neq 1$, by solution of systems of linear equations. Using the work of Section 2 we can construct a greater number of linear equations than the number o coefficients we desire to compute, as we shall see. The excess equations are then available as a check on the accuracy of the procedure. The relevan equations are (2.1) with $n = 0, 1, \ldots$, (2.11), (2.12), and the normalization relatior

$$\sum_{k=0}^{\infty} (-)^k E(k) = 1. \tag{4.1}$$

Suppose that we desire to obtain N unknowns, $E(0)$, $E(1)$, $\ldots$, $E(N-1)$ (To simplify the notation, we designate the computed values of $E(n)$ by $E(n)$) Three selections of systems of linear equations were studied. They are as follows.

Method A:

Equation (2.12)
Equations (2.1) with $n = 0, 1, \ldots, N-4$
Equation (2.11) with $E(N+k) = 0$ for $k \geqslant 0$
Equation (4.1) with $E(N+k) = 0$ for $k \geqslant 0$.

Method B:

Same as Method A except delete equation (4.1) as above and use insteac equation (2.1) with $n = N-3$ and $E(N)$ set to zero.

Method C:

Same as Method A except delete equations (2.11) and (4.1) and use insteac equations (2.1) with $n = N-3$ and $n = N-2$ and both $E(N)$ and $E(N+1)$ set to zero.

Use of (4.1) as a check is very much preferred as each coefficient is giver equal weight. From this point of view, Method A is the least preferred.

Now let

$$\gamma = \left| \sum_{k=0}^{N-1} (-)^k E(k) - 1 \right|, \tag{4.2}$$

nd take $z = 8$, $\mu = \nu = 0$. Then for $a = 0.8$, the values of γ for Methods B nd C were found as $1.04 \cdot 10^{-13}$ and $2.00 \cdot 10^{-15}$, respectively, while for $a = 0.9$, ıe values of γ for Methods B and C were determined as $1.88 \cdot 10^{-13}$ and $.00 \cdot 10^{-15}$, respectively. It appears that Method C is superior.

Finally, if $a = 1.0$ and $\mu = \nu = 0$, the coefficients $E(n)$ are readily deterıined once we know $C(n)$ for $\nu = 0$ and $\nu = 1$. This follows since [10]

$$\int_0^x e^{-t} I_0(t)\,dt = x e^{-x} [I_0(x) + I_1(x)]. \tag{4.3}$$

'hus, from (1.10), (1.11) and (2.5), we have

$$E(0) = C(0) + (z/8)[2D(0) + D(1)],$$

$$E(n+1) = C(n+1) + (z/8)\left[\frac{2}{\varepsilon_n} D(n) + 2D(n+1) + D(n+2)\right], \quad n = 0, 1, \ldots, \tag{4.4}$$

/here in the notation of (1.11) $C(n)$ is $C(n)$ for $\nu = 0$ and $D(n)$ is $C(n)$ for $\nu = 1$.

Generating a solution of a recursion system by solving linear equations was ıvestigated by Olver [11] for a second order nonhomogeneous difference equation.

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APPENDIX

Let

$$I_\nu(x) = \frac{(x/2)^\nu e^x}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} C(n) T_n^*(x/z), \tag{A.1}$$

$$\int_0^x e^{-at} t^\mu I_\nu(t)\,dt = \frac{x^{\mu+\nu+1} e^{-(a-1)x}}{2^\nu(\mu+\nu+1)\Gamma(\nu+1)} \sum_{n=0}^{\infty} E(n) T_n^*(x/z). \tag{A.2}$$

Iere we tabulate the coefficients $C(n)$ for $\nu = 0$ and $\nu = 1$, and the coeffiients $E(n)$ for $\mu = \nu = 0$ and $a = 0(0.1)1.0$. In each case $z = 8.0$. The oefficients were obtained as explained in the main body of the report. Where oefficients were found by more than one method, we note those which are ıe more accurate. The coefficients are accurate to at least 14 decimals, and re record all coefficients found to be greater in magnitude than $1.0 \cdot 10^{-16}$.

Kansas City (Missouri), September 1969.

TABLE I

CHEBYSHEV COEFFICIENTS FOR $I_\nu(x)$

$$I_\nu(x) = \frac{(x/2)^\nu e^x}{\Gamma(\nu+1)} \sum_{n=0}^{\infty} C(n)\, T_n^*(x/8), \qquad 0 \leqslant x \leqslant$$

n	$C(n),\ \nu = 0$		n	$C(n),\ \nu = 1$	
0	0.3383976372047393D	00	0	0.2525871864436334D	00
1	−0.3046826723431997D	00	1	−0.3528330367156680D	00
2	0.1716209015222096D	00	2	0.2052873173796943D	00
3	−0.9490109704804826D	−01	3	−0.1058919624161900D	00
4	0.4930528423967103D	−01	4	0.4945289806125308D	−01
5	−0.2373741480589963D	−01	5	−0.2112816978925243D	−01
6	0.1054646039459507D	−01	6	0.8312845888625792D	−02
7	−0.4324309995050611D	−02	7	−0.3027144901262513D	−02
8	0.1639475616941352D	−02	8	0.1024571912337155D	−02
9	−0.5763755745385886D	−03	9	−0.3235216316517945D	−03
10	0.1885028850958439D	−03	10	0.9563130215100145D	−04
11	−0.5754195010082168D	−04	11	−0.2654632731207898D	−04
12	0.1644844807072911D	−04	12	0.6940502616275388D	−05
13	−0.4416738358458809D	−05	13	−0.1713744052939098D	−05
14	0.1117387539120119D	−05	14	0.4006589507104288D	−06
15	−0.2670793853940650D	−06	15	−0.8890118257592696D	−07
16	0.6046995022542002D	−07	16	0.1876307477299165D	−07
17	−0.1300025009986268D	−07	17	−0.3774499503445684D	−08
18	0.2659823724682432D	−08	18	0.7251180563104290D	−09
19	−0.5189795601635347D	−09	19	−0.1332697944700418D	−09
20	0.9675809035373413D	−10	20	0.2347237259778203D	−10
21	−0.1726826291441584D	−10	21	−0.3967948795529930D	−11
22	0.2955052663129697D	−11	22	0.6447586731891222D	−12
23	−0.4856446783112030D	−12	23	−0.1008437100945595D	−12
24	0.7676185498605090D	−13	24	0.1520136858947103D	−13
25	−0.1168533287799370D	−13	25	−0.2211193895470806D	−14
26	0.1715391285555170D	−14	26	0.3107263915472447D	−15
27	−0.2431279846547990D	−15			

TABLE II

CHEBYSHEV COEFFICIENTS FOR $\int_0^x e^{-at} I_\nu(t)\,dt$

$$\int_0^x e^{-at} I_0(t)\,dt = x\,e^{(1-a)x} \sum_{n=0}^{\infty} E(n)\,T_n^*(x/8), \qquad 0 \leqslant x \leqslant 8$$

$E(n),\ a = 0$	n	$E(n),\ a = 0.1$
0.2355142644165745D 00	0	0.2437643472720758D 00
−0.3561696908894129D 00	1	−0.3617936463989773D 00
0.2158038198733890D 00	2	0.2131124433171958D 00
−0.1110609759496666D 00	3	−0.1063614371805490D 00
0.5007885950893066D −01	4	0.4662048799889590D −01
−0.2023648757728595D −01	5	−0.1841276154990959D −01
0.7457195231519404D −02	6	0.6676642784517917D −02
−0.2539360352686669D −02	7	−0.2251960877961192D −02
0.8067850643551902D −03	8	0.7124886871320533D −03
−0.2407510650299854D −03	9	−0.2125182260048852D −03
0.6777871043695278D −04	10	0.5993558221433477D −04
−0.1805660947817064D −04	11	−0.1601154357905868D −04
0.4561697299192887D −05	12	0.4057307216525433D −05
−0.1094718978098230D −05	13	−0.9764604675679276D −06
0.2499321606909110D −06	14	0.2234917577166657D −06
−0.5436469300347570D −07	15	−0.4871600767882978D −07
0.1128276749656779D −07	16	0.1012803878989169D −07
−0.2237419802937164D −08	17	−0.2011291748398009D −08
0.4245594307168707D −09	18	0.3820957086310229D −09
−0.7719789015476186D −10	19	−0.6954344485491595D −10
0.1346945592890375D −10	20	0.1214361161847740D −10
−0.2258160331193327D −11	21	−0.2037251081247612D −11
0.3642334128164151D −12	22	0.3287895642108522D −12
−0.5659283488746175D −13	23	−0.5111089446662262D −13
0.8480309416795369D −14	24	0.7662131398197879D −14
−0.1226925195609240D −14	25	−0.1108968177933504D −14
0.1715711555518740D −15	26	0.1551277875633633D −15

TABLE II (Continued)

n	$E(n),\ a = 0.2$	n	$E(n),\ a = 0.3$
0	0.2532164573035312D 00	0	0.2642163568415312D 00
1	— 0.3674260440624920D 00	1	— 0.3728828311773270D 00
2	0.2095297695160582D 00	2	0.2048289757762005D 00
3	— 0.1011596144361857D 00	3	— 0.9544560235121093D — 01
4	0.4310589058071759D — 01	4	0.3958726413288567D — 01
5	— 0.1668223844763907D — 01	5	— 0.1507120193537280D — 01
6	0.5977139634334146D — 02	6	0.5362016486595025D — 02
7	— 0.2005898322298745D — 02	7	— 0.1798278559629989D — 02
8	0.6344352070568645D — 03	8	0.5702885813804734D — 03
9	— 0.1896654681464313D — 03	9	— 0.1711383654044611D — 03
10	0.5366605719229450D — 04	10	0.4860489980396966D — 04
11	— 0.1438488521887995D — 04	11	— 0.1306975252399369D — 04
12	0.3656062845852603D — 05	12	0.3330242214262385D — 05
13	— 0.8820990915817417D — 06	13	— 0.8050719053487277D — 06
14	0.2023026360012976D — 06	14	0.1849183756448280D — 06
15	— 0.4416832957330181D — 07	15	— 0.4042084810295206D — 07
16	0.9194374917267067D — 08	16	0.8422201768272234D — 08
17	— 0.1827764610732366D — 08	17	— 0.1675536729013718D — 08
18	0.3475221400348688D — 09	18	0.3187775879951435D — 09
19	— 0.6329482602810542D — 10	19	— 0.5808990715403421D — 10
20	0.1105892239976350D — 10	20	0.1015402205628281D — 10
21	— 0.1856198673833137D — 11	21	— 0.1704965139488764D — 11
22	0.2996973268791583D — 12	22	0.2753708446043263D — 12
23	— 0.4660571489951227D — 13	23	— 0.4283518103626790D — 13
24	0.6989023564579576D — 14	24	0.6425244480907150D — 14
25	— 0.1011838026002280D — 14	25	— 0.9304298502135772D — 15
26	0.1415770825633001D — 15	26	0.1302131187874246D — 15

TABLE II (Continued)

n	$E(n)$, $a = 0.4$		n	$E(n)$, $a = 0.5$	
0	0.2772629684569658D	00	0	0.2930986626624389D	00
1	— 0.3778364083616925D	00	1	— 0.3817052868925602D	00
2	0.1987325953816506D	00	2	0.1909167038970861D	00
3	— 0.8924027305968265D	— 01	3	— 0.8261496148739359D	— 01
4	0.3613249462574757D	— 01	4	0.3282456863813299D	— 01
5	— 0.1360452819858672D	— 01	5	— 0.1230258497589666D	— 01
6	0.4831467505335241D	— 02	6	0.4381996368143783D	— 02
7	— 0.1625131465132256D	— 02	7	— 0.1481665520073823D	— 02
8	0.5176652359703381D	— 03	8	0.4742879792388477D	— 03
9	— 0.1560037434954495D	— 03	9	— 0.1434798250955767D	— 03
0	0.4445916790812110D	— 04	10	0.4100430516567792D	— 04
1	— 0.1198639555571029D	— 04	11	— 0.1107719585264034D	— 04
2	0.3060200164685091D	— 05	12	0.2832239022130507D	— 05
3	— 0.7408833834715285D	— 06	13	— 0.6864525257906271D	— 06
4	0.1703672508296083D	— 06	14	0.1579862443019701D	— 06
5	— 0.3727314481401579D	— 07	15	— 0.3458803552633175D	— 07
6	0.7771854283528098D	— 08	16	0.7215996684400041D	— 08
7	— 0.1547054227807479D	— 08	17	— 0.1437070444584170D	— 08
8	0.2944760721325348D	— 09	18	0.2736477988237306D	— 09
9	— 0.5368355785893131D	— 10	19	— 0.4990315697438658D	— 10
0	0.9387105693505430D	— 11	20	0.8728578479896087D	— 11
1	— 0.1576673490529644D	— 11	21	— 0.1466434482989383D	— 11
2	0.2547183958298218D	— 12	22	0.2369611383791517D	— 12
3	— 0.3963194425434013D	— 13	23	— 0.3687628373115547D	— 13
4	0.5946007095781493D	— 14	24	0.5533538315093795D	— 14
5	— 0.8611935235787061D	— 15	25	— 0.8015786704181582D	— 15
6	0.1205438355231707D	— 15	26	0.1122151776208979D	— 15

TABLE II (Continued)

n	$E(n),\ a = 0.6$		n	$E(n),\ a = 0.7$	
0	0.3128638087812015D	00	0	0.3383674127398192D	00
1	—0.3834527737800924D	00	1	—0.3812145312062837D	00
2	0.1810397060745701D	00	2	0.1688296233579173D	00
3	—0.7571769681751579D	—01	3	—0.6880243773329704D	—01
4	0.2975678663501961D	—01	4	0.2702085661309696D	—01
5	—0.1117733497457166D	—01	5	—0.1022833733210717D	—01
6	0.4006292581126654D	—02	6	0.3693814031049528D	—02
7	—0.1362680597306218D	—02	7	—0.1263092478227618D	—02
8	0.4381308218656933D	—03	8	0.4075241077288225D	—03
9	—0.1329481047926720D	—03	9	—0.1239449877856332D	—03
10	0.3807503806455274D	—04	10	0.3555307694416853D	—04
11	—0.1030137813067325D	—04	11	—0.9630166541617590D	—05
12	0.2636814160906564D	—05	12	0.2467165321840843D	—05
13	—0.6396337602221747D	—06	13	—0.5988912319749748D	—06
14	0.1473104653696412D	—06	14	0.1380032972705211D	—06
15	—0.3226843434297640D	—07	15	—0.3024334999456035D	—07
16	0.6735108526399231D	—08	16	0.6314806323085175D	—08
17	—0.1341809964919234D	—08	17	—0.1258474792462043D	—08
18	0.2555906293374229D	—09	18	0.2397819339411269D	—09
19	—0.4662312443852536D	—10	19	—0.4374967460544896D	—10
20	0.8156829696217365D	—11	20	0.7655677980131292D	—11
21	—0.1370667328703971D	—11	21	—0.1286685259812961D	—11
22	0.2215273093148515D	—12	22	0.2079871650749260D	—12
23	—0.3448013971385807D	—13	23	—0.3237723103352746D	—13
24	0.5174744194490159D	—14	24	0.4859757203482291D	—14
25	—0.7497037960784503D	—15	25	—0.7041495409371090D	—15

TABLE II (Continued)

n	$E(n),\ a = 0.8$	n	$E(n),\ a = 0.9$
0	0.3725711427009594D 00	0	0.4204718526451761D 00
1	— 0.3716004108518089D 00	1	— 0.3483695088261680D 00
2	0.1543031320250229D 00	2	0.1382724979787264D 00
3	— 0.6224654700787828D — 01	3	— 0.5651303055554610D — 01
4	0.2468502690372881D — 01	4	0.2276319098225341D — 01
5	— 0.9440256968103016D — 02	5	— 0.8784563497598108D — 02
6	0.3432255865607127D — 02	6	0.3209733300826150D — 02
7	— 0.1178454130200828D — 02	7	— 0.1105302220987000D — 02
8	0.3811927487031034D — 03	8	0.3582189752919951D — 03
9	— 0.1161364850291706D — 03	9	— 0.1092845397532936D — 03
10	0.3335427006801496D — 04	10	0.3141763466663757D — 04
11	— 0.9042906506932803D — 05	11	— 0.8524328621768029D — 05
12	0.2318367751206385D — 05	12	0.2186723128452493D - - 05
13	— 0.5630908110561696D — 06	13	— 0.5313714670244674D — 06
14	0.1298135648824127D — 06	14	0.1225491557134273D — 06
15	— 0.2845940907241380D — 07	15	— 0.2687557673072865D — 07
16	0.5944217169195313D — 08	16	0.5614950204825537D — 08
17	— 0.1184941362211844D — 08	17	— 0.1119566221079401D — 08
18	0.2258238481140575D — 09	18	0.2134077925822652D — 09
19	— 0.4121123445339843D — 10	19	— 0.3895221376413038D — 10
20	0.7212743255492820D — 11	20	0.6818417580619945D — 11
21	— 0.1212421335675160D — 11	21	— 0.1146292430288980D — 11
22	0.1960010970105059D — 12	22	0.1853328422263581D — 12
23	— 0.3050610241128809D — 13	23	— 0.2884870218833127D — 13
24	0.4569955627646220D — 14	24	0.4322022235981444D — 14
25	— 0.6523238729221414D — 15	25	— 0.6169211735720593D — 15

TABLE II (Concluded)

n	$E(n),\ a = 1.0$
0	0.4907389733763378D 00
1	—0.2998870555075745D 00
2	0.1234705371497399D 00
3	—0.5194480643948073D—01
4	0.2119094815673475D—01
5	—0.8228010434525622D—02
6	0.3016837481331716D—02
7	—0.1041181996612690D—02
8	0.3379529087013545D—03
9	—0.1032156233540217D—03
10	0.2969753043397334D—04
11	—0.8062799957702758D—05
12	0.2069381938261810D—05
13	—0.5030648973511869D—06
14	0.1160602050259531D—06
15	—0.2545972506249849D—07
16	0.5320417692030683D—08
17	—0.1061056277451965D—08
18	0.2022905393875645D—09
19	—0.3692872019540804D—10
20	0.6465092283726431D—11
21	—0.1087029234504579D—11
22	0.1757775038834536D—12
23	—0.2737205672172873D—13
24	0.4109688174962718D—14
25	—0.5956256879170446D—15

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orward Computation in Second)rder Difference Equations†

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ommunicated by R. P. Gilbert

this paper, we study the stability of certain computational algorithms associated with cond order difference equations. The methods are based on the forward use of the differce equation, and thus complement the known results for the Miller algorithm, which uses difference equation in the backward direction.

INTRODUCTION

omputation of sequences which satisfy difference equations has been dissed by a number of authors, such as J. C. P. Miller [1], F. W. J. Olver [2], Oliver [3], W. Gautschi [4], and the present author [5, 6]. These authors ve been primarily concerned with the properties of the Miller algorithm. is algorithm, which is due to J. C. P. Miller [1], uses the difference equation the backward direction. There has been less attention devoted to computan which utilizes the difference equation in the forward direction, not because forward algorithm is more difficult to analyze, but rather for the opposite ason—that its analysis was considered straightforward. Of the above thors, Oliver is the only one who studies extensively the stability properties algorithms based on the forward use of the difference equation.

It is the purpose of the present paper to point out that there are some therto unobserved aspects of forward computation with second order

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difference equations which can be exploited in practical applications. F instance, we show that for certain kinds of second order difference equation arbitrary rounding errors introduced in the course of the computations w not affect the relative error in the calculation of the desired solution, eve though these errors grow unboundedly as the computations proceed. Durir the course of our analysis, we are able to formulate our hypotheses in tern of the behavior of only one, rather than both, of the linearly independe solutions of the related homogeneous difference equation. This formulatic distinguishes our analysis from that of Oliver and also poses the problem in way that facilitates drawing conclusions about the algorithm.

2. THE ALGORITHM

The difference equation we shall consider is

$$y_k + a_k y_{k+1} + b_k y_{k+2} = h_k, \quad k = 0, 1, 2, \ldots, \; b_k \neq 0.$$

If this equation is used to compute y_k, starting with y_0 and y_1, it is to expected that certain rounding errors, δ_k, will inevitably be introduced in t course of the computations. What will actually be computed, then, is not but y_k^*, where y_k^* satisfies

$$y_k^* + a_k y_{k+1}^* + b_k y_{k+2}^* = h_k + \delta_k, \quad k = 0, 1, 2, \ldots$$

Let

$$\varepsilon_k = y_k - y_k^*,$$

be the *error* of the algorithm. Then

$$\varepsilon_k + a_k \varepsilon_{k+1} + b_k \varepsilon_{k+2} = -\delta_k, \quad k = 0, 1, 2, \ldots$$

Of importance in our analysis will be the related homogeneous equation

$$\phi_k + a_k \phi_{k+1} + b_k \phi_{k+2} = 0, \quad k = 0, 1, 2, \ldots$$

We shall prove the

THEOREM *Let there exist a solution of the homogeneous equation (5), u_k su that $u_k \neq 0$, $k = 0, 1, 2, \ldots$, and such that*

$$\left| \frac{b_k u_{k+2}}{u_k} \right|^{-1} \leqq A_k, \quad k = 0, 1, 2, \ldots$$

Then there exists a constant $c > 0$ such that

$$|\varepsilon_k| \leqq c|u_k| \, |V_k| \left\{ |\varepsilon_0| + |\varepsilon_1| + \sup_{0 \leqq j \leqq k-2} \left| \frac{\delta_j}{u_j} \right| \right\}, \quad k \geqq 2$$

where

$$V_k = \prod_{j=0}^{k-2} (1 + A_j) - 1$$

Proof Given that u_k satisfies (5), the complete solution of (4) may be tained by variation of parameters. One assumes that $\varepsilon_k = u_k v_k$. After bstitution in (4), it is found that Δv_k satisfies a first order difference equation ich may be solved completely. We omit details, but the result is that

$$\varepsilon_k = u_k \left\{\frac{\varepsilon_0}{u_0} + \left(\frac{\varepsilon_1}{u_1} - \frac{\varepsilon_0}{u_0}\right) \sum_{s=0}^{k-1} f_s + g_k\right\}, \quad k = 0, 1, 2, \ldots, \tag{9}$$

$$g_k = -\sum_{s=1}^{k-1} f_s \sum_{r=0}^{s-1} \delta_r / u_r f_r, \quad k = 0, 1, 2, \ldots, \tag{10}$$

$$f_k = (-)^k \sum_{j=0}^{k-1} \left(1 + a_j \frac{u_{j+1}}{u_j}\right)^{-1}, \quad k = 0, 1, 2, \ldots, \tag{11}$$

d empty sums are interpreted as zero, empty products as unity. We have

$$\frac{f_s}{f_r} = (-)^{s+r} \sum_{j=r}^{s-1} \left(1 + a_j \frac{u_{j+1}}{u_j}\right)^{-1}, \quad 0 \leqq r \leqq s. \tag{12}$$

it

$$\left|\left(1 + a_j \frac{u_{j+1}}{u_j}\right)\right|^{-1} = \left| -b_j \frac{u_{j+2}}{u_j}\right|^{-1} \leqq A_j. \tag{13}$$

e then find that

$$\begin{aligned} |g_k| &\leqq \sup_{0 \leqq j \leqq k-2} \left|\frac{\delta_j}{u_j}\right| \sum_{s=1}^{k-1} \sum_{r=0}^{s-1} A_r A_{r+1} \ldots A_{s-1} \\ &\leqq \sup_{0 \leqq j \leqq k-2} \left|\frac{\delta_j}{u_j}\right| V_k, \quad k \geqq 2. \end{aligned} \tag{14}$$

we let $\delta_r = 0, r > 0, \delta_0 = -u_0 f_0$ in (10), we get the first sum in (9) so (14) n be used also to majorize that sum. The result is

$$\begin{aligned} |\varepsilon_k| \leqq |u_k| \Bigg\{ &\left|\frac{\varepsilon_0}{u_0}\right| + \left(\left|\frac{\varepsilon_1}{u_1}\right| + \left|\frac{\varepsilon_0}{u_0}\right|\right) |f_0| V_k + \\ &\sup_{0 \leqq j \leqq k-2} \left|\frac{\delta_j}{u_j}\right| V_k \Bigg\}, \quad k \geqq 2. \end{aligned} \tag{15}$$

nce $V_k \neq 0$, it may be factored out of the quantity in parenthesis and the sult may be written as (7).

We now consider an application of this theorem to the following homo-neous difference equation

$$\begin{aligned} &y_k + \frac{(2k+v+3-a)\{(2k+v+2-a)(2k+v+4-a) - z(v-a)\}}{(k+1)(k+v+1-a)(2k+v+4-a)z^2} y_{k+1} - \\ &\frac{(2k+v+2-a) y_{k+2}}{(k+1)(k+v+1-a)(2k+v+4-a)z^2} = 0, \quad k = 0, 1, 2, \ldots, z \neq 0 \end{aligned} \tag{16}$$

which was introduced by Luke in his study of approximations to the inco plete and complete gamma functions [7, 8]. A class of rational approximatic to the incomplete gamma function

$$\nu\gamma(\nu, z) = \nu \int_0^z e^{-t}\, t^{\nu-1}\, dt = \frac{C_k(\nu, z)}{D_k(\nu, z)} + L_k(\nu, z) \qquad (\text{:}$$

were discussed in these references. C_k and D_k, which are polynomials in can be shown to satisfy (16). Another solution of (16) is, clearly, D_k Results in [8] give

$$D_k L_k = e^{-z/2}(-)^k \sqrt{\pi}\left(\frac{z}{2}\right)^{2k+\nu+1-a} k^{-\frac{1}{2}}[1+0(k^{-1})],\ k \to \infty, \qquad (1$$

$$D_k = \frac{\Gamma(2k+\nu+1-a)}{\Gamma(\nu+1)}\, e^{-z/2}[1+0(k^{-1})]. \qquad (1$$

From (17), (18), we see C_k behaves asymptotically the same as D_k except f a multiplicative term involving z.

Let $u_k = D_k$, which we assume is non-zero. Then

$$b_k \frac{u_{k+2}}{u_k} = -\frac{16k^2}{z^2}[1+0(k^{-1})], \qquad (2$$

so we can determine a constant $M > 0$ such that

$$\left| b_k \frac{u_{k+2}}{u_k} \right|^{-1} \leqq \frac{|z|^2 M}{16(k+1)^2} = A_k, \quad k = 0, 1, 2, \ldots \qquad (2$$

We see that V_k approaches a limit as $k \to \infty$, since the product in (8) co verges. The result is that we can determine a constant d such that

$$\left| \frac{\varepsilon_k}{D_k} \right| \leqq d \left\{ |\varepsilon_0| + |\varepsilon_1| + \sup_{0 \leqq j \leqq k-2} \left| \frac{\delta_j}{D_j} \right| \right\}, \quad k \geqq 2. \qquad (2$$

Thus the relative error involved in calculating D_k (or likewise, C_k) in t forward direction from equation (16) will be small if ε_0, ε_1 and sup $|\delta_j/D_j|$ a all small. In fact, the rounding error δ_k can grow as rapidly in k $\Gamma(2k+\nu+1-a)$ and the relative error will still remain bounded.

A useful concept computationally is that of the *relative* rounding errc $|\delta_j/u_j|$. We can prove the following simple corollary for the homogeneo equation (5) which relates the relative rounding error to the relative err in the calculated solution of the equation.

ɔROLLARY: Let u_k be a solution of (5) such that $u_k \neq 0$, $k = 0, 1, 2, \ldots$ and ch that

$$\left| b_k \frac{u_{k+2}}{u_k} \right|^{-1} \leqq M/(k+1)^\sigma, \quad k = 0, 1, 2, \ldots, \tag{23}$$

some $M > 0$, $\sigma > 1$. Then the relative error in the computation of u_k from in the forward direction cannot exceed a constant times the sum of the tial errors and the maximum relative roundoff error, i.e.,

$$\left| \frac{\varepsilon_k}{u_k} \right| \leqq d' \left\{ |\varepsilon_0| + |\varepsilon_1| + \sup_j \left| \frac{\delta_j}{u_j} \right| \right\}. \tag{24}$$

Proof It is necessary only to note that the product in (8) converges when $= M/(k+1)^\sigma$. Thus V_k is bounded.

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DERIVATIVE-FREE ITERATION PROCESSES*

JET WIMP†

1. Introduction. Let $\psi(y)$ be analytic in the neighborhood of $y = \alpha$, and

$$\psi(y) = \alpha + \sum_{r=1}^{\infty} c_r z^r, \quad z = y - \alpha, \quad |z| < \rho, \quad \rho > 0, \quad c_1 \neq 0, \tag{1}$$

so that the equation

$$\psi(x) = x \tag{2}$$

has a simple root at $x = \alpha$. Then a function $\Phi(\psi, y)$ which is analytic in a neighborhood of α,

$$\Phi = \alpha + \sum_{r=0}^{\infty} d_r z^{m+r}, \quad |z| < \sigma, \quad \sigma > 0, \quad d_0 \neq 0, \quad m \geqq 1, \tag{3}$$

is called *an iteration function of order m* (see [1]). Under certain conditions on Φ and ψ we would expect the sequence y_k defined by

$$y_{k+1} = \Phi(y_k), \quad k \geqq 0, \tag{4}$$

to have the property

$$\lim_{k \to \infty} y_k = \alpha, \tag{5}$$

provided that y_0 is sufficiently close to α.

Classical examples of iteration functions are

$$\Phi = \frac{y\psi'(y) - \psi(y)}{\psi'(y) - 1}, \tag{6}$$

which gives the Newton–Raphson iteration procedure and

$$\Phi = \frac{y\psi_2(y) - \psi(y)^2}{\psi_2(y) - 2\psi(y) + y}, \tag{7}$$

where we have used the notation (as we shall throughout)

$$\psi_0(y) = y, \quad \psi_{r+1}(y) = \psi_r[\psi(y)], \qquad r \geqq 0. \tag{8}$$

Equation (7) gives the Steffensen iteration procedure [2, p. 241]. Both (6) and (7) are of second order.

Several methods are known for constructing high order iteration processes. Schröder's method [3, p. 128] and more recently the method due to Kogan [4] provide iteration processes of arbitrary order $m = 2, 3, 4, \cdots$ which involve higher derivatives of ψ. They are both generalizations of Newton's method and reduce to that when $m = 2$.

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In this paper we seek to extend Steffensen's iteration method to iteration functions of arbitrary order which are rational in ψ_j, $0 \leqq j \leqq n$. We note that although iteration functions of higher order can be obtained from two functions Φ_1, Φ_2 of orders m_1 and m_2, respectively, by forming $\Phi_1(\Phi_2)$, the resulting function being of order $m_1 m_2$, it is clear that if we start with (7) for Φ_1 and Φ_2, the result is not rational in ψ_j and, further, the iteration functions so generated can only furnish methods of orders 2^k, $k \geqq 1$.

In this paper we give two methods for generating derivative-free iteration functions of arbitrary order $m = 2, 3, 4, \cdots$. When $m = 2$, both methods yield the Steffensen iteration function.

Throughout we use the notation $|c_{i,j}|_N$ for the Nth order determinant which contains in its ith row and jth column the element $c_{i,j}$ and the following notation for the Nth order alternate,

$$V_N(x_1, x_2, \cdots, x_N) = |x_j^{i-1}|_N = \prod_{j=1}^{N-1} \prod_{r=j+1}^{N} (x_r - x_j). \tag{9}$$

2. Preliminary result.

LEMMA. *Let*

$$D_m = |b_i(x_j)|_m, \tag{10}$$

where

$$b_i(x_j) = \sum_{s=0}^{\infty} a_{i-1,s} x_j^s, \quad 0 \leqq |x_j| < \rho_j, \quad \rho_j > 0. \tag{11}$$

Let

$$x_j = O(\eta), \quad \eta \to 0. \tag{12}$$

Then

$$D_m = |a_{i-1,j-1}|_m V_m(x_1, \cdots, x_m) + O[\eta^{m(m-1)/2+1}], \qquad m \geqq 1. \tag{13}$$

Proof. In what follows it will be convenient to let D_m be a generic notation not necessarily involving the same $b_i(x_j)$ wherever the symbol appears.

Proof is by induction. Assume (13) true for $1 \leqq m \leqq N - 1$. We see by (9) that this implies

$$D_m = O(\eta^{m(m-1)|2}), \qquad 1 \leqq m \leqq N-1. \tag{14}$$

$$D_N = |c_i(x_j)|_N + R_N, \tag{15}$$

$$c_i(x_j) = \sum_{s=0}^{N-1} a_{i-1,s} x_j^s = b_i(x_j) - d_i(x_j), \tag{16}$$

$$d_i(x_j) = \sum_{s=N}^{\infty} a_{i-1,s} x_j^s. \tag{17}$$

We have

$$|c_i(x_j)|_N = |a_{i-1,j-1}|_N |x_j^{i-1}|_N = |a_{i-1,j-1}|_N V_N(x_1, \cdots, x_N). \tag{18}$$

The remainder R_N may be written

$$R_N = \sum_{r=1}^{N} \sum_{(u_1, u_2, \cdots, u_r) \in {}_N S_r} (x_{u_1} x_{u_2} \cdots x_{u_r})^N T_N(u_1, u_2, \cdots, u_r), \tag{19}$$

where ${}_n S_k$ is the set of combinations of the first n integers taken k at a time, and T_N is a determinant of order N containing $d_i(x_{u_j})/x_{u_j}^N$ in the columns u_j, $j = 1, 2, \cdots, r$, and $c_i(x_k)$ in the kth column if $k \neq u_j$. By Laplace's expansion [5, p. 78], $T_N(u_1, u_2, \cdots, u_r)$ may be expanded by minors chosen from the columns u_j and their cofactors whose elements are chosen from the remaining $N - r$ columns. These latter are determinants of the form D_{N-r}, and each may be estimated as $\eta \to 0$ by (14). Thus (19) may be written

$$\begin{aligned} R_N &= \sum_{r-1}^{N} O\{\eta^{rN} \cdot \eta^{(N-r)(N-r-1)/2}\} \\ &= \sum_{r=1}^{N} O\{\eta^{N(N-1)/2 + r(r+1)/2}\} \\ &= O\{\eta^{N(N-1)/2+1}\}. \end{aligned} \tag{20}$$

This establishes the lemma for $m = N$. Since the result is true for $m = 1$, the proof is complete.

3. First process. Heuristically, we may think of the present iteration function as arising when the unknowns A_s are eliminated from the following system of equations:

$$\begin{aligned} &(\psi_j - \Phi) + \sum_{s=0}^{n-1} A_s(\psi_{s+j} - \psi_{s+j+1}) = 0, && 0 \leqq j \leqq n, \\ &\psi_0 = y, \quad \psi_{r+1} = \psi_r(\psi), && r \geqq 0. \end{aligned} \tag{21}$$

The determinant for the elimination is

$$\begin{aligned} &|v_{i,j}|_{n+1} = 0, \\ &v_{i,1} = \psi_{i-1} - \Phi, \quad v_{i,j} = \psi_{i+j-3} - \psi_{i+j-2}, && j > 1. \end{aligned} \tag{22}$$

Clearly (22) defines an iteration process, since (21) implies $\Phi(\alpha) = \alpha$. Elementary row and column manipulations give

$$\begin{aligned} &\Phi = \alpha + \frac{|\Delta_{i-1}(\psi_{j-1})|_{n+1}}{|\Delta_{i+1}(\psi_{j-1})|_n}, \\ &\Delta_{k+1}(y) = \Delta_k(\psi) - \Delta_k(y), \quad k \geqq 0, \quad \Delta_0(y) = z = y - \alpha. \end{aligned} \tag{23}$$

Now $\Delta_k(y)$ may be written

$$\Delta_k(y) = \sum_{r=1}^{\infty} \mu_{k,r} z^r, \qquad k \geqq 0, \tag{24}$$

where the $\mu_{k,r}$'s are computed recursively from the c_r's by means of the last part of (23) and (1). We have, for instance,

$$
\begin{aligned}
&\mu_{11} = (c_1 - 1), \quad \mu_{1j} = c_j, && j > 1,\\
(25)\qquad &\mu_{21} = (c_1 - 1)^2, \quad \mu_{22} = c_2(c_1 - 1)(c_1 + 2), \cdots, &&\\
&\mu_{31} = (c_1 - 1)^3, \quad \mu_{32} = c_2(c_1 - 1)^2(c_1^2 + 3c_1 + 3), \cdots . &&
\end{aligned}
$$

We now prove the following theorem.

THEOREM 1. *Let $|\mu_{i+1,j}|_n, |\mu_{i,j+1}|_n$ be nonzero, $c_1^r \neq c_1^s, r \neq s$. Then the iteration process*

$$(26)\qquad y_{k+1} = \Phi(y_k),$$

where $\Phi(y)$ is defined by (22), *is of order $n + 1$.*

Proof. For the numerator determinant in (23) we use the lemma with the identifications

$$(27)\qquad a_{i-1,s} = \mu_{i-1,s+1}, \quad x_j = \bar{\psi}_{j-1} = \psi_{j-1} - \alpha, \quad m = n + 1, \quad \eta = z,$$

and for the denominator determinant we let

$$(28)\qquad a_{i-1,s} = \mu_{i+1,s+1}, \quad x_j = \bar{\psi}_{j-1}, \quad m = n, \quad \eta = z.$$

The result is

$$(29)\qquad \Phi = \alpha + \bar{\psi}_n \frac{[|\mu_{i-1,j}|_{n+1} V_{n+1}(\bar{\psi}_0, \cdots, \bar{\psi}_n) + O(z^{n(n+1)/2+1})]}{[|\mu_{i+1,j}|_n V_n(\bar{\psi}_0, \cdots, \bar{\psi}_{n-1}) + O(z^{(n-1)/2+1})]}.$$

Since $|\mu_{i-1,j}|_{n+1} = |\mu_{i,j+1}|_n$, and

$$(30)\qquad \psi_r = \alpha + c_1^r z + O(z^2), \qquad r \geqq 0,$$

equation (29) becomes

$$(31)\qquad \Phi = \alpha + \frac{|\mu_{i,j+1}|_n}{|\mu_{i+1,j}|_n} c_1^{n(n+1)/2} \prod_{j=1}^{n} (c_1^j - 1) z^{n+1}[1 + O(z)].$$

This proves the theorem.

4. Second process. Here we eliminate the constants B_s from the system of equations:

$$(32)\qquad (\psi_j - \Phi) + \sum_{s=0}^{n-1} B_s(\psi_j - \psi_{j+1})^{s+1} = 0, \qquad 0 \leqq j \leqq n.$$

The eliminant is

$$(33)\qquad |\omega_{i,j}|_{n+1} = 0, \quad \omega_{i,1} = \psi_{i-1} - \Phi, \quad \omega_{i,j} = (\psi_{i-1} - \psi_i)^{j-1}, \qquad j > 1.$$

Straightforward algebra gives

$$
\begin{aligned}
&\Phi = \alpha + |\bar{\omega}_{i,j}|_{n+1}/V_{n+1}(\psi_0 - \psi_1, \psi_1 - \psi_2, \cdots, \psi_n - \psi_{n+1}), &&\\
(34)\qquad &\bar{\omega}_{i,1} = \psi_{i-1} - \alpha, \quad \bar{\omega}_{i,j} - \omega_{i,j}, && j > 1.
\end{aligned}
$$

Let

$$z = (y - \psi) \sum_{s=0}^{\infty} v_s (y - \psi)^s. \tag{35}$$

The series for z above is obtained by reversion of (1). For example,

$$v_0 = \frac{1}{(1 - c_1)}, \quad v_1 = \frac{c_2}{(1 - c_1)^3}, \quad v_2 = \frac{c_3}{(1 - c_1)^4} + \frac{2c_2}{(1 - c_1)^5}, \cdots. \tag{36}$$

We now apply the lemma to (34) with $x_j = \psi_{j-1} - \psi_j$, $\eta = z$,

$$a_{j-1,s} = \begin{cases} v_s, & j = 1, \\ \delta_{j-2,s}, & j > 1, \end{cases} \tag{37}$$

$$|a_{i-1,j-1}|_{n+1} = (-)^n v_n, \tag{38}$$

$\delta_{i,j}$ being the Kronecker delta. Then

$$\Phi = \alpha + \prod_{j=1}^{n+1} (\psi_{j-1} - \psi_j)[(-)^n v_n + O(z)] \tag{39}$$

provided the coefficient of $z^{n(n+1)/2}$ in $V_{n+1}(\psi_0 - \psi_1, \psi_1 - \psi_2, \cdots, \psi_n - \psi_{n+1})$ is not zero. We can characterize this condition by using (30) and (9) and we are led to the following theorem.

THEOREM 2. *Let* Φ, v_n *be defined by* (34), (35), *respectively. Let* $v_n \neq 0$ *and* $c_1^r \neq c_1^s$, $r \neq s$. *Then the iteration process*

$$y_{k+1} = \Phi(y_k) \tag{40}$$

is of order $n + 1$.

As an example of the two processes given, let $n = 2$. From Theorem 1 we have

(41)

$$\Phi = \frac{y(\psi_2\psi_4 - \psi_3^2) - \psi(\psi\psi_4 - \psi_2\psi_3) + \psi_2(\psi\psi_3 - \psi_2^2)}{(\psi_4 - 3\psi_3 + 3\psi_2 - \psi)(\psi_2 - 2\psi + y) - (\psi_3 - 3\psi_2 + 3\psi - y)(\psi_3 - 2\psi_2 + \psi)},$$

while Theorem 2 gives

$$\Phi = y + \frac{(y - \psi)}{(y - 2\psi_2 + \psi_3)} \left\{ \frac{(y - \psi)(\psi_2 - \psi_3)}{y - 2\psi + \psi_2} - \frac{(y - \psi_2)(\psi - \psi_2)}{y - \psi - \psi_2 + \psi_3} \right\}. \tag{42}$$

A comparison of the two shows that the latter is more efficient computationally since it requires fewer evaluations of ψ, and this is true in general. The process of Theorem 1 requires ψ_j for $1 \leqq j \leqq 2n$ and Theorem 2 requires ψ_j only for $1 \leqq j \leqq n + 1$.

Note added in proof. The reader will observe that the iteration scheme defined by (23) is formally related to the general $(n + 1)$th order sequence-to-sequence transformation given by Shanks (J. Math. Phys., 34 (1955)). For $n = 1$, the comparison furnished is between Steffensen's iteration process and Aitken's δ^2 procedure.

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RECENT DEVELOPMENTS IN RECURSIVE COMPUTATION*

JET WIMP†

1. Introduction. In this work, we shall give a treatment of the Miller algorithm and other related methods which have been devised to generate sequences satisfying linear difference equations. We shall confine our attention to those sequences which satisfy the general second order nonhomogeneous difference equation

$$\mathfrak{A}_n(y(n)) = h(n), \tag{1.1}$$
$$\mathfrak{A}_n \equiv I + a_1(n)E + a_2(n)E^2, \qquad a_2(n) \neq 0, \quad n = 0, 1, 2, \cdots,$$

where I is the identity and E the shift operator, $Ey(n) = y(n + 1)$.

A common feature of all the methods we shall discuss is that they make use of the difference equation in the backward direction; i.e., one solves the equation for $y(n)$ in terms of $h(n)$, $y(n + 1)$, $y(n + 2)$ and, for a large integer m beginning with arbitrary values $y(m + 2) = \alpha$, $y(m + 1) = \beta$, $|\alpha| + |\beta| \neq 0$, one calculates successively $y(m)$, $y(m - 1)$, $\cdots$, $y(0)$ from the equation. Each of these methods considered attempts to approximate the given sequence by $y(n)$ or by a linear combination of such sequences, hopefully with arbitrarily high accuracy as $m \to \infty$.

The simplest and historically the first of these computational methods was developed by J. C. P. Miller in 1954 to compute modified Bessel functions of integer order. This method, the Miller algorithm, proceeded as follows:

Consider the difference equation

$$y(n) - \frac{2(n + 1)}{x} y(n + 1) - y(n + 2) = 0, \qquad x > 0, \quad n \geqq 0, \tag{1.2}$$

which is satisfied by the modified Bessel functions $I_n(x)$ and $(-)^n K_n(x)$. Let m be an integer $\geqq 0$. Put

$$\Lambda_{m+1}(m) = 0, \qquad \Lambda_m(m) = 1, \tag{1.3}$$

and calculate $\Lambda_n(m)$ for $0 \leqq n \leqq m - 1$ from (1.2), i.e.,

$$\Lambda_n(m) = \frac{2(n + 1)}{x} \Lambda_{n+1}(m) + \Lambda_{n+2}(m), \qquad 0 \leqq n \leqq m - 1. \tag{1.4}$$

Now the series

$$1 = \sum_{k=0}^{\infty} (-)^k \varepsilon_k I_{2k}(x), \qquad \varepsilon_k = \begin{cases} 1, & k = 0, \\ 2, & k > 0, \end{cases} \tag{1.5}$$

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is known (Erdélyi et al. [5, vol. II, p. 7]).

Let

$$\Omega(m) = \sum_{k=0}^{[m/2]} (-)^k \varepsilon_k \Lambda_{2k}(m), \tag{1.6}$$

where $[m/2]$ means the largest integer not greater than $m/2$. Then, by using the known asymptotic properties of I_n, K_n for large n, one can show that

$$\lim_{m \to \infty} \Lambda_n(m)/\Omega(m) = I_n(x), \qquad n \geqq 0, \quad x > 0. \tag{1.7}$$

In fact, the asymptotic estimates

$$I_n(x) = \frac{(x/2)^n}{n!}[1 + O(n^{-1})], \qquad (-)^n K_n(x) = \frac{(-2/x)^n}{2}\Gamma(n)[1 + O(n^{-1})] \tag{1.8}$$

follow from the ascending series representations of I_n and K_n, and, since $\Lambda_n(m)$ satisfies (1.2), it can be represented as a linear combination of the (linearly independent) solutions (1.8); see the Appendix. This means that

$$\Lambda_n(m) = c_1(m) I_n(x) + c_2(m)(-)^n K_n(x). \tag{1.9}$$

From (1.3) and

$$I_m(x) K_{m+1}(x) + I_{m+1}(x) K_m(x) = 1/x, \tag{1.10}$$

we conclude that

$$c_1(m) = x K_{m+1}(x), \qquad c_2(m) = x(-)^m I_{m+1}(x). \tag{1.11}$$

Thus

$$\begin{aligned} \Lambda_n(m) &= m!(2/x)^m I_n(x)[1 + O(m^{-1})], && m \to \infty, \\ \Omega(m) &= m!(2/x)^m [1 + O(m^{-1})], && m \to \infty, \end{aligned} \tag{1.12}$$

so (1.7) follows.

The above analysis shows clearly why the process converges, and also why it converges to I_n and not to $(-)^n K_n$: I_n is very small compared to K_n as $n \to \infty$. This characteristic of Miller's algorithm, namely, that the solution of the difference equation to which the process converges, if it converges, must, in a certain sense be the smallest solution, remains true when the algorithm is applied to general homogeneous difference equations.

A remarkable feature of the Miller algorithm is that no tabular values of I_n are needed in the computations, only a normalization relationship, such as (1.5). Tabular values would be required, of course, if (1.2) were used in the forward direction, and moreover, when (1.2) is used in the forward direction to compute I_n starting with initial values of I_0 and I_1, those small errors inevitably introduced in the course of the computation grow rapidly with n. Such a phenomenon is called instability.[1]

[1] As the reader can verify, the computation of $(-)^n K_n$ by using (1.2) in the *forward* direction with initial values of K_0 and K_1 is stable, i.e., the relative error due to random errors which are introduced during the computations does not grow with n. In general, a difference equation can be used efficiently in the forward direction only to compute the "largest" solution of the equation. However, the analysis of the forward procedure is rather less of a problem than the analysis of Miller's algorithm (see Gautschi [8]) and will occupy none of our attention here.

The method proposed by Miller created enormous interest, and a number of papers subsequently appeared in which the writers either further treated the application of the method to Bessel functions, or else showed that the method could be used to compute other special functions. Stegun and Abramowitz [25], Randels and Reeves [22], Goldstein and Thaler [11], Corbató and Uretsky [4], and Makinouchi [13]–[15], all treated the computation of Bessel functions. Rotenberg [23] showed how the algorithm could be used to compute toroidal harmonics (i.e., Legendre functions) and Miller himself applied the method to parabolic cylinder functions [17].

Gautschi [6] discussed the computation of repeated integrals of the error function

$$y(n) = i^n \operatorname{erfc} x = (2/\sqrt{\pi} n!) \int_x^\infty (t - x)^n e^{-t^2} dt, \qquad n \geqq 0, \tag{1.13}$$

which satisfy

$$y(n) = 2xy(n + 1) - 2(n + 2)y(n + 2) = 0, \tag{1.14}$$

and in a later paper [7] discussed the computation by backward recursion of a number of other functions defined by definite integrals.

The Miller algorithm can be applied to problems other than the computation of the special functions. Recently, it has been employed in such diverse problems as the calculation of successive derivatives of $[f(z)/z]$, where f is an arbitrary analytic function (Gautschi [9]) and the computation of coefficients for the Chebyshev polynomial expansions of functions which satisfy differential equations with polynomial coefficients (Clenshaw [1], [2], Clenshaw and Picken [3], Luke [12]). Olver [19]–[21] and Gautschi [7], [10] have analyzed the error properties of the algorithm for general second order difference equations.

Although in this work we treat only algorithms based on the second order difference equation, we remind the reader that many of the methods we give have been developed in some generality for difference equations of higher order, particularly by Gautschi [8], Wimp [27], Oliver [18], Luke [12].

We provide proofs only of results which are new or which are not readily available in the literature, for instance, results in the thesis of Wimp [27].

Throughout, certain basic properties of linear difference equations are used. These are contained in the Appendix.

2. Algorithms.

2.1. The homogeneous case, $h(n) \equiv 0$. Here the equation of interest is

$$\mathfrak{A}_n(y(n)) = 0, \qquad n = 0, 1, 2, \cdots. \tag{2.1}$$

We will first discuss the classical Miller algorithm. A good treatment of this algorithm is given by Gautschi [10] who has many examples. Valuable additional material and generalizations are given by Olver [19], [21], Oliver [18], Wimp [27]. The algorithm proceeds as follows. Put

$$\Lambda_{m+1}(m) = 0, \qquad \Lambda_m(m) = 1 \tag{2.2}$$

and

$$\mathfrak{A}_n(\Lambda_n(m)) = 0, \qquad 0 \leqq n \leqq m-1. \tag{2.3}$$

Let $y_1(n) \not\equiv 0$ be the solution of the homogeneous equation we wish to compute, and suppose we are given the *normalization relationship*

$$1 = \sum_{k=0}^{\infty} L(k) y_1(k). \tag{2.4}$$

Let

$$\Omega(m) = \sum_{k=0}^{m} L(k) \Lambda_k(m), \tag{2.5}$$

$$\Gamma_n(m) = \Lambda_n(m)/\Omega(m). \tag{2.6}$$

(Notice it is no loss of generality to assume the sum (2.4) is 1, since $L(k)$ can always be so normalized.) The Miller algorithm is then described by the following.

THEOREM 1. *Let there exist a solution of* (2.1), $y_2(n)$, *which is linearly independent of* $y_1(n)$, *with the property that* $y_2(n)$ *is not zero for* n *sufficiently large, and*

$$\lim_{n\to\infty} \frac{y_1(n+1)}{y_2(n+1)} \sum_{k=0}^{n} L(k) y_2(k) = \lim_{n\to\infty} \frac{y_1(n)}{y_2(n)} = 0. \tag{2.7}$$

Then

$$\lim_{m\to\infty} \Gamma_n(m) = y_1(n), \qquad n = 0, 1, 2, \cdots. \tag{2.8}$$

Proof. See the cited references.

In the special case where $y_1(0) \neq 0$ is known, we may take $L(0) = 1/y_1(0)$ and the theorem simplifies to the following.

COROLLARY 1. *Let* $y_2(n)$ *be as in Theorem* 1 *but with* (2.7) *amended to*

$$\lim_{n\to\infty} \frac{y_1(n)}{y_2(n)} = 0. \tag{2.9}$$

Then

$$\lim_{m\to\infty} \frac{\Lambda_n(m)}{\Lambda_0(m)} y_1(0) = y_1(n), \qquad n = 0, 1, 2, \cdots. \tag{2.10}$$

A solution of (2.1), $y(n) \not\equiv 0$, having the property

$$\lim_{n\to\infty} y(n)/y^*(n) = 0, \tag{2.11}$$

where $y^*(n)$ is any other solution of (2.1) which is not a constant multiple of $y(n)$, is called by Gautschi a *minimal solution*, by Wimp an *antidominant solution*. It is, apart from a multiplication constant, unique. Clearly the $y_1(n)$ of Theorem 1 is a minimal solution. An alternative formulation of the corollary is that *if* (2.1) *possesses a minimal solution* $y(n)$ *with* $y(0) \neq 0$ *then* $\Lambda_n(m)/\Lambda_0(m)$ *will converge as* $m \to \infty$ *to a constant multiple of* $y(n)$.

In practical applications the asymptotic behavior of the desired solution $y_1(n)$ may be known, say from an integral representation, while the behavior of the other member of the basis, $y_2(n)$, may not. The following theorem may then be useful.

THEOREM 2. *Let* (2.2)–(2.6) *hold and let* $y_1(n) \neq 0$, $n \geqq n_0 \geqq 0$ *for some* n_0. *Let*

(2.12) $$\lim_{n\to\infty} S(n)^{-1} = 0,$$

(2.13) $$\lim_{n\to\infty} S(n)^{-1} \sum_{k=n_0+1}^{n} L(k)y_1(k)S(k-1) = 0,$$

where

(2.14) $$S(n) = \sum_{k=n_0}^{n} \mu(k), \qquad \mu(k) = \prod_{j=n_0}^{k-1} a_2(j)^{-1}/y_1(k)y_1(k+1).$$

Then

(2.15) $$\lim_{m\to\infty} \Gamma_n(m) = y_1(n), \qquad n = 0, 1, 2, \cdots.$$

Proof. For $n \geqq n_0$ we can define a solution of (2.1) which is linearly independent of $y_1(n)$ by

(2.16) $$y_2(n) = y_1(n)\{c_0 S(n-1) + c_1\}$$

and for $0 \leqq n \leqq n_0$ let $y_2(n)$ be calculated from (2.1). Note that

(2.17) $$c_0 = D_2(n_0), \qquad c_1 = y_2(n_0)/y_1(n_0)$$

(see the Appendix for the notation D_2).

Now $y_2(n_0)$, $y_2(n_0+1)$ are arbitrary except that c_0 must not be zero. Elementary inequalities then show that (2.12) and (2.13) imply the conditions (2.7) of Theorem 1. As an example, consider the original Miller algorithm for the calculation of I_n. We have

$$a_2(n) \equiv -1, \qquad y_1(n) \equiv I_n = \frac{(x/2)^n}{n!}[1 + O(n^{-1})],$$

(2.18) $$\mu(k) = (-)^{k+n_0}(2/x)^{2k+1}k!(k+1)![1 + O(k^{-1})],$$

$$S(n) = \mu(n)[1 + O(n^{-1})].$$

The requirement (2.12) is obviously satisfied. Furthermore, (2.13) is fulfilled by any $L(k)$ which makes (2.4) converge, since $L(k)y_1(k) = o(1)$ at most.

An analogue of Corollary 1 is the following.

COROLLARY 2. *Let* $y_1(0) \neq 0$, $y_1(n) \neq 0$ *for* $n \geqq n_0 \geqq 0$ *for some* n_0. *Let*

(2.19) $$|y_1(n)/a_2(n)y_1(n+2)| \geqq \delta > 2, \qquad n \geqq n_0.$$

Then

(2.20) $$\lim_{m\to\infty} \Lambda_n(m)y_1(0)/\Lambda_0(m) = y_1(n), \qquad n = 0, 1, 2, \cdots.$$

Proof. Let $y_2(n)$ be as defined by (2.16). We have

$$|y_2(n+1)| \geqq |y_1(n+1)|\left\{|c_0||\mu(n)| - |c_1| - |c_0| \sum_{k=n_0}^{n-1} |\mu(k)|\right\}. \tag{2.21}$$

But (2.19) can be written

$$|\mu(n+1)/\mu(n)| \geqq \delta, \qquad n \geqq n_0, \tag{2.22}$$

or

$$-|\mu(n)| \geqq -|\mu(n+1)|/\delta, \qquad n \geqq n_0. \tag{2.23}$$

Thus

$$\begin{aligned} |y_2(n+1)/y_1(n+1)| &\geqq \{|c_0||\mu(n)|[1 - \delta^{-1} - \delta^{-2} - \cdots - \delta^{n_0 - n}] - |c_1|\} \\ &\geqq \{|c_0|(\delta - 2)|\mu(n)|/(\delta - 1) - |c_1|\} > 0, \end{aligned} \tag{2.24}$$

for n sufficiently large, since $\mu(n) \to \infty$. We have

$$|y_1(n+1)/y_2(n+1)| \leqq ||c_0|(\delta - 2)|\mu(n)|/(\delta - 1) - |c_1||^{-1}, \tag{2.25}$$

so the above ratio vanishes as $n \to \infty$ and Corollary 1 can be invoked.

The preceding results apply only when the equation possesses a minimal solution. In those cases where the equation does not possess a minimal solution but a basis can be determined whose members exhibit similar behavior as $n \to \infty$, an ingenious algorithm first used by Clenshaw [2] is often applicable. The method assumes that we are given *two* different normalization relationships for the desired solution.

THEOREM 3. *Let $y_2(n)$ be nonzero for n sufficiently large. Define*

$$T_{t,r} \equiv T_{t,r}(m) = \sum_{k=0}^{m} L_t(k) y_r(k), \qquad r, t = 1, 2. \tag{2.26}$$

Let (2.2)–(2.6) *hold with* $L(k) \equiv L_1(k)$,

$$\lim_{m \to \infty} T_{t,1} = 1, \qquad t = 1, 2, \tag{2.27}$$

$$\lim_{m \to \infty} T_{t,2} = A_t, \qquad t = 1, 2, \quad A_1 \neq A_2. \tag{2.28}$$

Let $|y_1(n)/y_2(n)|$ be bounded and bounded away from zero as $n \to \infty$ and $\Omega(m)$ be nonzero for m sufficiently large. Then for m_1 sufficiently large we can determine $m_2 > m_1$ so that the equation

$$\mu \sum_{k=0}^{m_1} L_2(k)\Gamma_k(m_1) + (1 - \mu) \sum_{k=0}^{m_2} L_2(k)\Gamma_k(m_2) = 1 \tag{2.29}$$

has a unique solution, μ (depending, of course, on m_1 and m_2). Furthermore, let

$$-\frac{y_1(m_2+1)}{y_2(m_2+1)} + \frac{y_1(m_1+1)}{y_2(m_1+1)} \tag{2.30}$$

be bounded and bounded away from zero as $m_1 \to \infty$. Then

$$\lim_{m_1 \to \infty} \mu\Gamma_n(m_1) + (1 - \mu)\Gamma_n(m_2) = y_1(n), \qquad n = 0, 1, 2, \cdots. \tag{2.31}$$

The proof of this result (and its generalization to higher order difference equations) is given by Wimp [27].

Clenshaw [2] first employed the algorithm described by the above theorem to compute the coefficients in certain Chebyshev polynomial expansions.

If two values of $y_1(n)$ are known, the following result can be used.

COROLLARY 3. *Let* $|y_1(n)/y_2(n)|$ *be bounded and bounded away from zero as* $n \to \infty$. *We can determine* k_1 *and* k_2, $0 \leqq k_1 < k_2$, *so that*

$$y_1(k_1)y_2(k_2) - y_1(k_2)y_2(k_1) \neq 0, \tag{2.32}$$

and for m_1 *sufficiently large,* m_1 *and* $m_2 \geqq m_1$ *can then be determined so that the system of equations*

$$\sum_{v=1}^{2} \mu_v \Lambda_{k_j}(m_v)/\Lambda_{k_1}(m_v) = y_1(k_j), \qquad j = 1, 2, \tag{2.33}$$

has a unique solution (μ_1, μ_2).

Let (2.30) *be bounded and bounded away from zero. Then*

$$\lim_{m_1 \to \infty} \mu_1 \frac{\Lambda_n(m_1)}{\Lambda_{k_1}(m_1)} + \mu_2 \frac{\Lambda_n(m_2)}{\Lambda_{k_1}(m_2)} = y_1(n). \tag{2.34}$$

This result and its generalization are given by Wimp [27].

For a large class of second order difference equations, the algorithms above are in a certain sense exhaustive, i.e., at least one of them will converge to a solution of the equation. That class consists of the equation whose coefficients possess developments in Poincaré type asymptotic series

$$n^{r/\omega}\{c_0 + c_1 n^{-1/\omega} + c_2 n^{-2/\omega} + \cdots\}, \qquad c_0 \neq 0, \quad n \to \infty, \tag{2.35}$$

where r and ω are integers, $\omega > 1$, $r \geqq 0$. The above cited reference provides the following result.

THEOREM 4. *Let* $a_1(n)$ *and* $a_2(n)$ *possess asymptotic series of the kind* (2.35). *Then either*

$$\lim_{m \to \infty} \Lambda_n(m)/\Lambda_{n^*}(m) = y(n), \qquad n = 0, 1, 2, \cdots, \tag{2.36}$$

exists for $n^* = 0$ *or* 1, *is not identically zero and satisfies* (2.1) *or else* m_1 *and* m_2 *can be chosen so that* (2.29), (2.31) *hold for some solution* $y_1(n)$ *of* (2.1).

We have confined ourselves entirely to statements about the convergence of the Miller and Clenshaw algorithms and have left undiscussed such important questions as the numerical accuracy or efficiency of the algorithms, i.e., the behavior of the error incurred by taking a finite value of m. There has recently been a lot of interesting work done in this area by Olver and other writers, notably Tait [26]. There are also methods available for increasing the efficiency of the Miller algorithm, see Shintani [24] and Wimp [27]. From the former author we have an economical method of computing $\Omega(m)$, $\Lambda_n(m)$ embodied in the following theorem.

THEOREM 5. Ω *satisfies*

$$a_2(m)\Omega(m) + a_1(m+1)\Omega(m+1) + \Omega(m+2) = L(m+2), \qquad m \geqq 0, \tag{2.37}$$

with $\Omega(0) = L(0)$, $\Omega(1) = -a(0)L(0) + L(1)$ *and* $\Lambda_n(m)$ *satisfies*

$$a_2(m)\Lambda_n(m) + a_1(m+1)\Lambda_n(m+1) + \Lambda_n(m+2) = 0, \qquad m \geqq 0. \tag{2.38}$$

2.2. The nonhomogeneous case. The Miller algorithm can also be applied in a straightforward manner to the nonhomogeneous equation

$$\mathfrak{A}_n(y(n)) = h(n), \qquad h(n) \not\equiv 0, \quad n = 0, 1, 2, \cdots. \tag{2.39}$$

We have the following.

THEOREM 6. *Let m be an integer $\geqq 1$ and define the sequence $\Lambda_n(m)$ by*

$$\Lambda_{m+1}(m) = \Lambda_m(m) = 0, \tag{2.40}$$

$$\mathfrak{A}_n(\Lambda_n(m)) = h(n), \qquad 0 \leqq n \leqq m-1, \tag{2.41}$$

where $h(0) \neq 0$.

Let there exist a basis for (2.1), $[y_1(n), y_2(n)]$ *and a particular solution $p(n)$ of* (2.39) *which satisfy*

$$\lim_{n\to\infty} Y_r^*(n)h(n+1) = 0, \qquad r = 2, 3, \tag{2.42}$$

(*see* (A.14)). *Then*

$$\lim_{m\to\infty} \Lambda_n(m) = p(n), \qquad n = 0, 1, 2, \cdots. \tag{2.43}$$

Proof. We may write

$$\Lambda_n(m) = c_1(m)y_1(n) + c_2(m)y_2(n) + p(n). \tag{2.44}$$

Substituting (2.40) into (2.44) and solving, we obtain

$$c_1(m) = Y_3^*(m-1)h(m)/K, \qquad c_2(m) = Y_2^*(m-1)h(m)/K, \tag{2.45}$$

K as in (A.15), and this establishes (2.43).

From (2.45) and (A.13), (A.14) follows the important result

$$\mathfrak{B}_m^*(\Lambda_n(m+1)/h(m+1)) = 0, \tag{2.46}$$

which is useful for generating the $\Lambda_n(m)$'s systematically.

In the case where $p(0)$ is known from some source, an algorithm due to Olver [20] may be applied.

THEOREM 7. *Let there exist a basis of* (2.1), $[y_1(n), y_2(n)]$, *and a particular solution $p(n)$ of* (2.39) *which satisfy $y_1(0) \neq 0$, $y_2(n) \neq 0$ for n sufficiently large and*

$$\lim_{n\to\infty} y_1(n)/y_2(n) = \lim_{n\to\infty} p(n)/y_2(n) = 0. \tag{2.47}$$

Then the solution of the system of equations

$$\begin{aligned} \mathfrak{A}_n(\Lambda_n(m)) &= h(n), \qquad 0 \leqq n \leqq m-1; \\ \Lambda_0(m) = p(0), &\qquad \Lambda_{m+1}(m) = 0 \end{aligned} \tag{2.48}$$

possesses the property

$$\lim_{m\to\infty} \Lambda_n(m) = p(n), \qquad n = 0, 1, 2, \cdots. \tag{2.49}$$

Olver shows how the tridiagonal system (2.48) may be easily solved by simple elimination followed by back substitution.

Olver's algorithm often succeeds when the method formulated in Theorem 6 fails. As an example, let us take

$$a_1(n) = -\frac{2(n+1)}{x}, \quad a_2(n) = -1, \quad h(n) = (-)^n L(n), \tag{2.50}$$

where $L(n)$ is chosen so that

$$\begin{aligned} &L(n+1)/L(n) = O(1), \qquad n \to \infty, \\ &L(n)/L(n+1) = O(1), \qquad n \to \infty. \end{aligned} \tag{2.51}$$

Then

$$y_1(n) = I_n(x), \qquad y_2(n) = (-)^n K_n(x). \tag{2.52}$$

Furthermore, $p(n)$ as defined by

$$D_2(0)p(n) = -y_1(n)\sum_{k=0}^{n-1} y_2(k+1)L(k) - y_2(n)\sum_{k=n}^{\infty} y_1(k+1)L(k) \tag{2.53}$$

is easily shown to satisfy (2.39). If we use the asymptotic estimates (1.8) and (2.51), we find

$$p(n) = \frac{(-)^{n+1}L(n-1)}{2nD_2(0)}[1 + O(n^{-1})], \qquad n \to \infty. \tag{2.54}$$

Equation (2.42) is not satisfied for $r = 3$, but the conditions for the convergence of Olver's algorithm are. It is important to note that Olver's algorithm in this example provides a method of computing a solution of the nonhomogeneous equation (2.1) which is not necessarily a minimal solution of the related homogeneous equation (A.12). The same phenomenon occurs in nonhomogeneous difference equations of order higher than two, see Wimp and Luke [28]. Olver [20] has given also a form of Theorem 7 which utilizes a general normalization relation for $p(n)$, essentially the same algorithm given later by Wimp and Luke [28]. However, this algorithm, which still requires that $y_1(n)/y_2(n)$ and $p(n)/y_2(n)$ vanish as $n \to \infty$, can be considered a limiting case of the method to be given below, which has the extraordinary property that it imposes no conditions on the relative growth of the functions $y_1(n)$, $y_2(n)$ and $p(n)$. The algorithm, which requires two normalization conditions for the desired solution of (2.39), $p(n)$, proceeds as follows. First we take an integer $m \geqq 0$ and compute three sequences $H_n^{(r)}(m)$, $r = 1, 2, 3$, by

$$\mathfrak{A}_n(H_n^{(r)}(m)) = 0, \qquad 0 \leqq n \leqq m-1, \quad r = 1, 2, \tag{2.55}$$

$$\mathfrak{A}_n(H_n^{(3)}(m)) = h(n), \qquad 0 \leqq n \leqq m-1. \tag{2.56}$$

The initial values $H_{m+e}^{(r)}(m)$, $e = 0, 1$, $r = 1, 2, 3$, can be chosen arbitrarily, subject to a condition specified by (2.60) below. Now define a new sequence

$$\Lambda_n(m) = \sum_{r=1}^{3} c_r(m) H_n^{(r)}(m), \qquad c_3(m) \equiv 1, \tag{2.57}$$

where $c_1(m)$, $c_2(m)$ are chosen to satisfy the two equations

$$1 = \sum_{r=1}^{3} c_r(m) S_{t,r}(m), \qquad t = 1, 2, \tag{2.58}$$

with

$$S_{t,r} \equiv S_{t,r}(m) = \sum_{k=0}^{m} L_t(k) H_k^{(r)}(m). \tag{2.59}$$

The values $c_1(m)$, $c_2(m)$ satisfying (2.58) exist if

$$S_{1,1}S_{2,2} - S_{2,1}S_{1,2} \neq 0. \tag{2.60}$$

(This guarantees that the initial values of the sequences are chosen so that the sequences are linearly independent.)

We can now prove the following theorem.

THEOREM 8. *Let* (2.55)–(2.60) *hold.*

Let (2.1) *possess a basis* $[y_1(n), y_2(n)]$ *and* (2.39) *a particular solution* $p(n)$ *satisfying the conditions below*:

$$1 = \sum_{k=0}^{\infty} L_t(k) p(k), \qquad t = 1, 2, \tag{2.61}$$

$$\lim_{m \to \infty} (T_{1,3}T_{2,2} - T_{2,3}T_{1,2})/U = \lim_{m \to \infty} (T_{1,1}T_{2,3} - T_{2,1}T_{1,3})/U = 0, \tag{2.62}$$

$$U = T_{1,1}T_{2,2} - T_{2,1}T_{1,2} \neq 0, \tag{2.63}$$

m sufficiently large, where

$$T_{t,r} \equiv T_{t,r}(m) = \sum_{k=0}^{m} L_t(k) y_r(k), \qquad t, r = 1, 2, \tag{2.64}$$

$$T_{t,3} \equiv T_{t,3}(m) = \sum_{k=m+1}^{\infty} L_t(k) p(k) < \infty, \qquad r = 1, 2. \tag{2.65}$$

Then

$$\lim_{m \to \infty} \Lambda_n(m) = p(n), \qquad n = 0, 1, 2, \cdots. \tag{2.66}$$

Proof. We may write

$$\Lambda_n(m) = d_1(m) y_1(n) + d_2(m) y_2(n) + p(n) \tag{2.67}$$

and imposing the conditions (2.58) provides two equations for $d_1(m)$, $d_2(m)$. The resulting expressions for these quantities are seen to vanish as $m \to \infty$ provided the conditions of the theorem are satisfied.

Of course, the values of m chosen for $H_n^{(1)}$, $H_n^{(2)}$, $H_n^{(3)}$ need not be the same. Let $m_1 \leqq m_2 \leqq m_3$ and choose c_1, c_2 to satisfy the two equations

$$1 = \sum_{r=1}^{3} c_r S_{t,r}(m_r), \qquad t = 1, 2, \quad c_3 \equiv 1. \tag{2.68}$$

Then define the new sequence

$$\Lambda_n = \sum_{r=1}^{3} c_r H_n^{(r)}(m_r). \tag{2.69}$$

We would expect that, under certain conditions,

$$\lim_{m_1 \to \infty} \Lambda_n = p(n), \qquad n = 0, 1, 2, \cdots. \tag{2.70}$$

This algorithm is a generalization of both the method of Theorem 8 and of Clenshaw's algorithm. The conditions for its validity, while straightforward in derivation, are complicated and we defer further remarks to a future paper.

We can see that the algorithm defined in Theorem 8 does not, inherently, impose any relative growth conditions on the functions $y_1(n)$, $y_2(n)$, $p(n)$, by taking the simple case where the $L_t(k)$ are such that *all* the series (2.59) converge. Let

$$\beta_{t,r} = \lim_{m \to \infty} T_{t,r}(m), \qquad t, r = 1, 2. \tag{2.71}$$

Then if

$$\beta_{11}\beta_{22} - \beta_{21}\beta_{12} \neq 0 \tag{2.72}$$

it is found that

$$\lim_{m \to \infty} d_1(m) = \lim_{m \to \infty} d_2(m) = 0 \tag{2.73}$$

and the algorithm converges.

Convergence alone does not mean that the algorithm is satisfactory from the practical point of view. Wimp and Luke [28] for a third order difference equation treat the special limiting case when the above method has only one normalization relation and find that there occurs a calamitous accumulation of roundoff error as a certain parameter in the difference equation approaches a limiting value. Nevertheless, the algorithm converges theoretically. The problem is caused by the fact that the $L_t(k)H_k^{(r)}(m)$ are large and alternate in sign as $k = 0, 1, 2, \cdots$. As Wimp and Luke show, such instabilities can often be overcome by solving the system of linear equations

$$\begin{aligned} &\mathfrak{A}_n(\Lambda_n(m)) = h(n), && 0 \leqq n \leqq m - 1, \\ &\sum_{k=0}^{m+1} L_t(k)\Lambda_k(m) = 1, && t = 1, 2, \end{aligned} \tag{2.74}$$

rather than generating the sequences $H_n^{(r)}(m)$ directly from the difference equations and forming $\Lambda_n(m)$ from (2.57). Since, as F. W. J. Olver has pointed out to the author, the use of Theorem 8 will often cause problems because of errors due to cancellation in applying equation (2.57), and since extremely efficient procedures for solving systems of equations are available on today's computers, the above algorithm is often to be preferred to Theorem 8.

Appendix. We are interested in both the homogeneous equation

$$\mathfrak{A}_n(y(n)) = 0, \tag{A.1}$$

and in the nonhomogeneous equation

$$\mathfrak{A}_n(y(n)) = h(n), \qquad h(n) \not\equiv 0. \tag{A.2}$$

Corresponding to any given initial conditions $y(k_1)$, $y(k_2)$ both (A.1) and (A.2) possess a unique solution for all $n \geqq 0$. For the homogeneous equation, two linearly independent solutions can be determined, i.e., solutions $[y_1(n), y_2(n)]$ for which

$$c_1 y_1(n) + c_2 y_2(n) = 0, \qquad n = 0, 1, 2, \cdots, \tag{A.3}$$

implies $c_1 = c_2 = 0$. These are called a *basis*. Any other solution of (A.1) may be written

$$y(n) = d_1 y_1(n) + d_2 y_2(n), \tag{A.4}$$

d_1, d_2 constant. In what follows, $[y_1(n), y_2(n)]$ will denote a basis of (A.1). Any solution of (A.2) has the form

$$y(n) = e_1 y_1(n) + e_2 y_2(n) + p(n), \tag{A.5}$$

where $p(n)$ is a particular solution of (A.2).

The equation adjoint to (A.1) is

$$\begin{aligned} &\mathfrak{A}_n^*(y(n)) = 0, \\ &\mathfrak{A}_n^* \equiv a_2(n)I + a_1(n+1)E + E^2 \end{aligned} \tag{A.6}$$

and has the basis $[y_1^*(n), y_2^*(n)]$,

$$y_1^*(n) = -y_2(n+1)/D_2(n), \qquad y_2^*(n) = y_1(n+1)/D_2(n), \tag{A.7}$$

$$D_2(n) = y_1(n)y_2(n+1) - y_1(n+1)y_2(n). \tag{A.8}$$

Let

$$D_3(n) = \begin{vmatrix} y_1(n) & y_2(n) & p(n) \\ y_1(n+1) & y_2(n+1) & p(n+1) \\ y_1(n+2) & y_2(n+2) & p(n+2) \end{vmatrix}. \tag{A.9}$$

Then

$$D_2(n) = D_2(0) \prod_{j=0}^{n-1} a_2(j)^{-1}, \qquad n = 0, 1, 2, \cdots, \tag{A.10}$$

$$D_3(n) = \frac{D_3(0)h(n)}{h(0)} \prod_{j=0}^{n-1} a_2(j+1)^{-1}, \qquad n = 0, 1, 2, \cdots, \tag{A.11}$$

where we have used the convention, to be retained throughout this paper, that empty products are one, empty sums zero.

The homogeneous equation

$$\begin{aligned} &\mathfrak{B}_n(y(n)) = 0, \qquad n = 0, 1, 2, \cdots, \\ &\mathfrak{B}_n \equiv I + b_1(n)E + b_2(n)E^2 + b_3(n)E^3, \\ &b_1(n) = a_1(n) - h(n)/h(n+1), \\ &b_2(n) = a_2(n) - a_1(n+1)h(n)/h(n+1), \\ &b_3(n) = -a_2(n+1)h(n)/h(n+1) \end{aligned} \tag{A.12}$$

has the basis $[y_1(n), y_2(n), p(n)]$, while its adjoint

$$\text{(A.13)} \qquad \begin{aligned} &\mathfrak{B}_n^*(y(n)) = 0, \qquad n = 0, 1, 2, \cdots, \\ &\mathfrak{B}_n^* \equiv b_3(n)I + b_2(n+1)E + b_1(n+2)E^2 + E^3, \end{aligned}$$

has the basis $[Y_1^*(n), Y_2^*(n), Y_3^*(n)]$,

$$\text{(A.14)} \qquad \begin{aligned} Y_1^*(n) &= (y_1(n+1)y_2(n+2) - y_1(n+2)y_2(n+1))/D_3(n), \\ Y_2^*(n) &= -(y_1(n+1)p(n+2) - y_1(n+2)p(n+1))/D_3(n), \\ Y_3^*(n) &= (y_2(n+1)p(n+2) - y_2(n+2)p(n+1))/D_3(n). \end{aligned}$$

From (3.8)–(3.11) we see

$$\text{(A.15)} \qquad Y_1^*(n) = K/h(n), \qquad K = D_2(0)h(0)/a_2(0)D_3(0).$$

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PAPER II–6

*Rational approximations to Tricomic's Ψ funtion**

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IN THIS PAPER we derive closed form rational approximations to the Tricomi Ψ function ([1]),

$$\Psi(a, c; v) = \frac{1}{\Gamma(a)} \int_0^{\infty e^{i\varphi}} e^{-vt} t^{a-1}(1 + t)^{c-a-1}\, dt,$$

$$\operatorname{Re} a > 0,\ |\arg (e^{i\varphi}v)| < \pi/2, \quad -\pi < \varphi < \pi, \tag{1}$$

which converge uniformly on compact subsets of the sector $|\arg v| < \pi/2$, $v \neq 0$. As Tricomi's Ψ function can be written in terms of the Meijer G-function ([1])

$$\Psi(a, c; v) = \frac{v^{-a}}{\Gamma(a)\Gamma(1 + a - c)} G^{1,2}_{2,1}\left(v^{-1} \middle| \begin{matrix} 1 - a, c - a \\ 0 \end{matrix}\right), \tag{2}$$

we actually develop rational approximations to the G-function

$$E(v) = G^{1,2}_{2,1}\left(v^{-1} \middle| \begin{matrix} 1 - \alpha_1, 1 - \alpha_2 \\ 0 \end{matrix}\right)$$

$$= \frac{1}{2\pi i} \int_L \Gamma(-s)\Gamma(s + \alpha_1)\Gamma(s + \alpha_2)\, v^{-s}\, ds, \quad |\arg v| < 3\pi/2, \tag{3}$$

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where the contour L runs from $-i\infty$ to $+i\infty$, and separates the poles of $\Gamma(-s)$ from those of $\Gamma(s+\alpha_1)\,\Gamma(s+\alpha_2)$, see [1]. If $\alpha_1-\alpha_2$ is not an integer, it follows from the residue theorem that

$$E(v) = \Gamma(\alpha_2-\alpha_1)\,\Gamma(\alpha_1)\,v^{\alpha_1}{}_1F_1\left(\left.\begin{matrix}\alpha_1\\ 1+\alpha_1-\alpha_2\end{matrix}\right| v\right)$$

$$+\,\Gamma(\alpha_1-\alpha_2)\,\Gamma(\alpha_2)\,v^{\alpha_2}{}_1F_1\left(\left.\begin{matrix}\alpha_2\\ 1+\alpha_2-\alpha_1\end{matrix}\right| v\right). \tag{4}$$

Our rational approximations for $E(v)$ are obtained as follows. If the contour L in (3) is moved k units to the right, we obtain

$$E(v) = \Gamma(\alpha_1)\,\Gamma(\alpha_2)\sum_{j=0}^{k-1}(\alpha_1)_j\,(\alpha_2)_j\,\frac{(-v)^{-j}}{j!} + R_k(v),$$

$$R_k(v) = \frac{(-v)^{-n}}{2\pi i}\int_L \frac{\Gamma(-s)\,\Gamma(s+1)\,\Gamma(s+n+\alpha_1)\,\Gamma(s+n+\alpha_2)}{\Gamma(s+1+n)}\,v^{-s}\,ds,$$

$$(\sigma)_\mu = \frac{\Gamma(\sigma+\mu)}{\Gamma(\sigma)}. \tag{5}$$

As it can be shown that $R_k(v) = O(v^{-k})$ as $v\to\infty$ and $|\arg v| < 3\pi/2$, (5) is a paraphrase of the statement

$$E(v) \sim \Gamma(\alpha_1)\,\Gamma(\alpha_2)\;{}_2F_0\left(\alpha_1,\alpha_2\middle| -\frac{1}{v}\right),$$
$$v\to\infty,\ |\arg v| < 3\pi/2. \tag{6}$$

Replacing k by $k+1$ in (5), multiplying the resulting equation by arbitrary $A_{n,k}\gamma^k$, and summing from $k=0$ to a fixed integer n, we obtain the equations

$$h_n(\gamma)\,E(v) = \psi_n(v,\gamma) + F_n(v,\gamma)$$

$$h_n(\gamma) = \sum_{k=0}^{n}\gamma^k A_{n,k},\quad F_n(v,\gamma) = \sum_{k=0}^{n}\gamma^k A_{n,k}R_{k+1}(v) \tag{7}$$

$$\psi_n(v,\gamma) = \sum_{k=0}^{n}\gamma^k A_{n,k}\Gamma(\alpha_1)\,\Gamma(\alpha_2)\sum_{j=0}^{k}(\alpha_1)_j\,(\alpha_2)_j\,\frac{(-v)^{-j}}{j!}.$$

Then we see that $\psi_n(v,\gamma)/h_n(\gamma)$ is a formal rational approximation to $E(v)$ and $F_n(v,\gamma)/h_n(\gamma)$ is its corresponding error. The triangular form $\psi_n(v,\gamma)$

can also be written as

$$\psi_n(v, \gamma) = \Gamma(\alpha_1)\Gamma(\alpha_2) \sum_{k=0}^{n} \sum_{j=0}^{n-k} A_{n,j+k} \frac{(\alpha_1)_j (\alpha_2)_j (\gamma)^k (-\gamma/v)^j}{j!}. \tag{8}$$

In Fields [2], the above formulation was shown to be equivalent to the Lanczos τ-method, see [3], and the following theorem was proved.

THEOREM 1 *If* $|\arg v| < \pi/2$, $v \neq 0$, *and*

$$h_n(\gamma) = {}_2F_2\left(\begin{matrix} -n, n + \lambda \\ 1 + \alpha_1, 1 + \alpha_2 \end{matrix} \middle| -\gamma\right), \lambda > 0, \tag{9}$$

then

$$\operatorname*{limit}_{n\to\infty} \frac{\psi_n(v, v)}{h_n(v)} = E(v), \quad \operatorname*{limit}_{n\to\infty} \frac{F_n(v, v)}{h_n(v)} = 0. \tag{10}$$

As the asymptotic estimate

$$\begin{aligned} {}_2F_2\left(\begin{matrix} -n, n + \lambda \\ 1 + \alpha_1, 1 + \alpha_2 \end{matrix} \middle| -v\right) &\sim \frac{\Gamma(1 + \alpha_1)\Gamma(1 + \alpha_2)}{2\pi\sqrt{3}} (n^2 v)^\tau \\ &\times \exp\left(3(n^2 v)^{1/3} - \frac{v}{3}\right)\{1 + O(n^{-2/3})\}, \end{aligned} \tag{11}$$

$$\tau = -(1 + \alpha_1 + \alpha_2)/3; \quad n \to \infty, \quad |\arg v| < \pi,$$

was already known, see [4], the proof of Theorem 1 reduced to obtaining a proper estimate for $F_n(v, v)$. This was effected by showing that the differential operator which annihilates $E(v)$,

$$\mathscr{H} = (\delta - \alpha_1)(\delta - \alpha_2) - v\delta, \quad \delta = v\frac{d}{dv}, \tag{12}$$

when applied to $F_n(v, \gamma)$ yields

$$\mathscr{H}\{F_n(v, \gamma)\} = -\sum_{k=0}^{n} A_{n,k} \frac{\Gamma(k + 1 + \alpha_1)\Gamma(k + 1 + \alpha_2)}{k!} \left(-\frac{\gamma}{v}\right)^k. \tag{13}$$

Thus, if the $A_{n,k}$ are chosen as indicated in (9), the right-hand side of (13) is essentially a Jacobi polynomial which has a uniform algebraic rate of growth in n, $O(n^\sigma)$, for $0 \leqq \gamma/v \leqq 1$. A variation of parameter's technique then implies

$$F_n(v, v) = O(n^\sigma), \quad n \to \infty, \quad v \text{ fixed}. \tag{14}$$

Note that in Theorem 1, the parameter λ is essentially unspecified. By specializing λ and considering difference instead of differential operators, we obtain, among other benefits, a more convenient formulation of the error $F_n(v, v)$.

Let

$$U(\mu, n, \lambda) = \frac{(n+\lambda-1)(n+\mu)}{2n+\lambda-1} E^0 - \frac{n(n+\lambda-1-\mu)}{2n+\lambda-1} E^{-1},$$
$$U^*(n, \lambda) = \operatorname*{limit}_{\mu\to\infty} \frac{U(\mu, n, \lambda)}{\mu}, \tag{15}$$

where E^{-j} is the shift operator on n, i.e., $E^{-j}\{f(n)\} = f(n-j)$, and

$$M(\gamma) = U(0, n, \lambda-2)\, U(\alpha_1, n, \lambda-1)\, U(\alpha_2, n, \lambda) - n(n+\lambda-3)\gamma E^{-1} U^*(n, \lambda),$$
$$= A_0\left[E^0 + \sum_{j=1}^{3}[A_j + \gamma B_j]\, E^{-j}\right], \quad A_0 = \frac{n(n+\alpha_1)(n+\alpha_2)(n+\lambda-3)_3}{(2n+\lambda-3)_3},$$
$$A_1 = \frac{(n-1)(2n+\lambda-2)_2(n+\alpha_1-1)(n+\alpha_2-1)}{(n+\lambda-1)(2n+\lambda-4)(n+\alpha_1)(n+\alpha_2)} - \frac{n(2n+\lambda-2)}{(n+\lambda-1)},$$
$$A_2 = \frac{(n-1)(2n+\lambda-1)(n+\lambda-\alpha_1-2)(n+\lambda-\alpha_2-2)}{(n+\lambda-1)(n+\alpha_1)(n+\alpha_2)}$$
$$-\frac{(n-1)(n+\lambda-3)(2n+\lambda-2)_2(n+\lambda-\alpha_1-3)(n+\lambda-\alpha_2-3)}{(n+\lambda-2)_2(2n+\lambda-5)(n+\alpha_1)(n+\alpha_2)},$$
$$A_3 = \frac{(n-2)_2(2n+\lambda-2)_2(n+\lambda-\alpha_1-3)(n+\lambda-\alpha_2-3)}{(2n+\lambda-5)_2(n+\lambda-2)_2(n+\alpha_1)(n+\alpha_2)},$$
$$B_1 = -\frac{(2n+\lambda-2)_2}{(n+\lambda-1)(n+\alpha_1)(n+\alpha_2)},$$
$$B_2 = -\frac{(n-1)(2n+\lambda-2)_2}{(n+\lambda-2)_2(n+\alpha_1)(n+\alpha_2)}, \quad B_3 = 0. \tag{16}$$

We then have

THEOREM 2 *If the $A_{n,k}$ are chosen so that*

$$h_n(\gamma) = {}_2F_2\left(\begin{matrix} -n, n+\lambda \\ 1+\alpha_1, 1+\alpha_2 \end{matrix}\middle| -\gamma\right),$$
$$\psi_n(v, \gamma) = \Gamma(\alpha_1)\Gamma(\alpha_2) \sum_{k=0}^{n}\sum_{j=0}^{n-k} \frac{(-n)_{k+j}(n+\lambda)_{k+j}(\alpha_1)_j(\alpha_2)_j(-\gamma)^k(\gamma/v)^j}{(1+\alpha_1)_{k+j}(1+\alpha_2)_{k+j}(k+j)!\,j!}, \tag{17}$$

then

$$M(\gamma)\{h_n(\gamma)\} = 0$$

$$M(\gamma)\{\psi_n(v,\gamma)\} = -n(n+\lambda-3)\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)(\gamma/v) \times {}_2F_1\left(\begin{matrix}-n+1, n+\lambda-2\\ 2\end{matrix}\middle|\frac{\gamma}{v}\right), \tag{18}$$

$$M(v)\{\psi_n(v,v)\} = (-1)^n \frac{\Gamma(n+\lambda-2)\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)}{\Gamma(n)\Gamma(\lambda-2)}.$$

Proof All these results follow directly by computation from the operator equations

$$\begin{aligned} U(\mu, n, \lambda)\{(-n)_s(n+\lambda)_s\} &= (-n)_s(n+\lambda-1)_s(s+\mu), \\ U^*(n,\lambda)\{(-n)_s(n+\lambda)_s\} &= (-n)_s(n+\lambda-1)_s. \end{aligned} \tag{19}$$

COROLLARY 2.1 *If in Theorem 2, $\lambda - 3$ is a negative integer, then*

$$M(v)\{F_n(v,v)\} = 0. \tag{20}$$

Hence, to analyze the error $F_n(v,v)$, in this case, it is sufficient to analyze the equation

$$M(v)\{g_n(v)\} = 0. \tag{21}$$

To do this, we introduce some recent results of Wimp, [5]. Let

$$\begin{aligned} g_n(w) &= \frac{\Gamma(n+1)}{\Gamma(n+\lambda)} G^{3,1}_{2,3}\left(w\middle|\begin{matrix}1-n-\lambda, n+1\\ 0, -\alpha_1, -\alpha_2\end{matrix}\right), \\ &= \frac{1}{2\pi i}\int_L \frac{\Gamma(-s)\Gamma(-s-\alpha_1)\Gamma(-s-\alpha_2)(n+\lambda)_s}{(n+1)_{-s}} w^s\,ds, \end{aligned} \tag{22}$$

then Wimp's work* shows that

$$g_n(w) \sim \frac{(2\pi)}{\sqrt{3}}[n^2w]^\tau \exp\left(-3[n^2w]^{1/3} + \frac{w}{3}\right)\{1 + O(n^{-1/3})\},$$
$$\tau = -(1+\alpha_1+\alpha_2)/3; \quad n \to +\infty, \quad |\arg w| < 3\pi/2. \tag{23}$$

---

* This is not quite the function that Wimp treated in [5]. But a close inspection of that work shows that his analysis is actually applicable. The identification process with Wimp's work is made by replacing his λ, n and γ by w, $n+\lambda$ and $-\lambda$, respectively. Also, the application of Lemma 3 in this reference is made easier by employing the fact $2d_1 - d_2 = 1 + \gamma$.

This leads to our main result,

THEOREM 3 *If* $|\arg v| < \pi/2$, $v \neq 0$, *then* $g_n(ve^{\pi i})$, $g_n(ve^{-\pi i})$ *and* $h_n(v)$ *as defined by (17) form a basis of soluti onsfor the difference equation (21).*

Proof It follows from (11) and (23) that the three functions are linearly independent as functions of n in the right half plane. A direct computation using the integral representation in (22) and an analog of (19) shows that $g_n(ve^{\pm\pi i})$ satisfy (21).

COROLLARY 3.1 *If*

$$h_n(v) = {}_2F_2\left(\begin{matrix} -n, n+\lambda \\ 1+\alpha_1, 1+\alpha_2 \end{matrix}\middle| -v\right), \quad \lambda = 1 \quad \text{or} \quad 2,$$

$$\psi_n(v, v) = \Gamma(\alpha_1)\Gamma(\alpha_2) \sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{(-n)_{k+j}(n+\lambda)_{k+j}(\alpha_1)_j(\alpha_2)_j(-v)^k}{(1+\alpha_1)_{k+j}(1+\alpha_2)_{k+j}(k+j)!\,j!} \tag{24}$$

and $|\arg v| < \pi/2$, $v \neq 0$, *then there exist well defined analytic functions* $C^-(v)$, $C^+(v)$ *independent of n such that*

$$\begin{aligned} &G^{1,2}_{2,1}\left(v^{-1}\middle|\begin{matrix} 1-\alpha_1, 1-\alpha_2 \\ 0 \end{matrix}\right) - \frac{\psi_n(v, v)}{h_n(v)} \\ &= C^-(v)\frac{g_n(ve^{-\pi i})}{h_n(v)} + C^+(v)\frac{g_n(ve^{+\pi i})}{h_n(v)} \\ &\sim \sum_{\varepsilon=+,-} \frac{C^\varepsilon(v)\,4\pi^2}{\Gamma(1+\alpha_1)\Gamma(1+\alpha_2)} \exp\left(-3\sqrt{3}\,(n^2 ve^{i\varepsilon\pi/2})^{1/3} + i\pi\varepsilon\tau\right) \\ &\times \{1 + O(n^{-1/3})\}, \quad n \to \infty. \end{aligned} \tag{25}$$

Proof Clearly, the left-hand side of (25) is just the error $F_n(v, v)/h_n(v)$. From Corollary 2.1 and Theorem 3 it follows that there exist functions $C^\varepsilon(v)$, $\varepsilon = +, -, 0$, in $|\arg v| < \pi/2$, such that

$$F_n(v, v) = C^+(v)\,g_n(ve^{+\pi i}) + C^-(v)\,g_n(ve^{-\pi i}) + C^0(v)\,h_n(v). \tag{26}$$

The functions $C^\varepsilon(v)$ can be found in theory by setting n equal to zero, one and then two in the first line of (25) and then solving the resulting equations. The linear independence of $g_n(ve^{-\pi i})$, $g_n(ve^{+\pi i})$ and $h_n(v)$ implies the analytic character of the $C^\varepsilon(v)$ in $|\arg v| < \pi/2$, $v \neq 0$. From Theorem 1, we deduce that $C^0(v)$ is identically zero in $|\arg v| < \pi/2$. Equation (26) then reduces to the first line of (25). The last line of (25) follows from the preceding asymptotic estimates and the simple fact, $\sqrt{3}e^{i\varepsilon\pi/6} = 1 + e^{\varepsilon\pi i/3}$, $\varepsilon = \pm 1$.

COROLLARY 3.2 The sequence of rational approximations in Corollary 3.1 converges uniformly to $G_{2,1}^{1,2}\left(v^{-1}\middle|\begin{matrix}1-\alpha_1, 1-\alpha_2\\ 0\end{matrix}\right)$ on compact subsets of $|\arg v| < \pi/2$, $v \neq 0$.

Finally, we reiterate the main advantages of the rational approximations in Corollary 3.1. First, they are explicit, as opposed to mini-max rational approximations which, in general, can only be given numerically. Second, they can be computed fairly easily, due to the fact that both numerator and denominator polynomials, $\psi_n(v, v)$ and $h_n(v)$, satisfy the same third order recursion relation. And last, an explicit form for the error is known.

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Jacobi series which converge to zero, with applications to a class of singular partial differential equations

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1. *Introduction*. Expansions in series of functions are one of the most important tools of the applied mathematician, particularly expansions in series of the classical orthogonal polynomials, e.g. Laguerre, Jacobi and Hermite polynomials. In applied problems, the uniqueness of the particular expansion is usually intrinsic to the analysis, and often implicitly assumed. Indeed, in those cases where the functions in the series are orthogonal, uniqueness can often be proved by an argument that runs as follows. Let $\{\phi_n(x)\}$ $(n = 0, 1, 2, \ldots)$ be a sequence of functions orthogonal with respect to the weight function $\rho(x)$ over the interval $[0, 1]$, and suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x), \tag{1}$$

$$= \sum_{n=0}^{\infty} d_n \phi_n(x), \tag{2}$$

the series being boundedly convergent for $0 \leqslant x \leqslant 1$.

Then

$$0 = \sum_{n=0}^{\infty} (c_n - d_n)\, \phi_n(x), \tag{3}$$

and multiplying this series by $\phi_m(x)\,\rho(x)$ and integrating between 0 and 1, which is permissible, see (1), we find

$$c_m = d_m, \quad (m = 0, 1, 2, \ldots). \tag{4}$$

Even when the $\{\phi_n(x)\}$ are not orthogonal, one can show, as above, that the problem of uniqueness involves the question of whether 0 has a non-trivial representation as a series of the functions in question.

It is perhaps too little understood that care must be exercised in assuming that such expansions are unique, even in the case of the classical orthogonal polynomials. For example, let

$$R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x-1) \tag{5}$$

be the shifted Jacobi polynomial, the notation on the right above, as all other notation here, being that of (2). We show in this paper that one can determine subsets of $[0, 1]$ of measure 1 where

$$0 = \sum_{n=0}^{\infty} c_n R_n^{(\alpha,\beta)}(x), \tag{6}$$

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yet $c_n \neq 0$ for every n. This result, which holds provided only that $\alpha < -\frac{1}{2}$ and $\beta \neq -1, -2, \ldots$, has important applications in other areas of mathematics. As an example we use it to prove that conditions which are shown to guarantee uniqueness of the solutions of a class of singular partial differential equations cannot be relaxed.

The phenomenon (6) is not confined to Jacobi series, for the above statement is a corollary of a result which holds for sequences $\{g_n(x)\}$ defined by a wide class of generating functions

$$G(x,t) = \sum_{n=0}^{\infty} g_n(x)\, t^n. \tag{7}$$

2. *Results.*

THEOREM 1. *Let*

$$\lim_{n\to\infty} n g_n(x) = 0 \quad (x \in X), \tag{8}$$

where $\{g_n(x)\}$ *is defined by* (7) *for* $|t| < 1$, *so that* $G(x,t)$ *in* (7) *is analytic for* $|t| < 1$. *Assume furthermore that for each* $x \in X$, G *is also analytic at* $t = 1$ *and satisfies*

$$\left.\frac{\partial G(x,t)}{\partial t}\right|_{t=1} = KG(x,1) \quad (K \neq 0). \tag{9}$$

Then

$$0 = \sum_{n=0}^{\infty} (n-K)\, g_n(x) \quad (x \in X). \tag{10}$$

Proof. By (3), the series (10) converges. We have

$$G(x,t) = G(x,1)\,[1+K(t-1)] + O[(t-1)^2] \quad (t \to 1); \tag{11}$$

so

$$\left.\begin{aligned} t\frac{d}{dt}G(x,t) - KG(x,t) &= K(t-1)\,G(x,1) + O[(t-1)^2], \\ &= \sum_{n=0}^{\infty} (n-K)\, g_n(x) t^n, \end{aligned}\right\} \tag{12}$$

and (10) follows by Abel's theorem (4).

We now consider the case where $\{g_n(x)\}$ are the shifted Jacobi polynomials.

In what follows, let

$$X_1 = (0,1), \quad X_2 = [0,1), \quad X_3 = [0,1], \quad \gamma = \alpha+\beta+1. \tag{13}$$

THEOREM 2. *Let* $\alpha < -\frac{1}{2}$, $\beta \neq -1, -2, \ldots$, *and*

(i) *if* $\gamma < 0$, *then* $r = 2$;

(ii) *if both* $\gamma < 0$, $\alpha < -1$, *then* $r = 3$;

(iii) $r = 1$ *if neither of the above prevails.*

Then

$$0 = \sum_{n=0}^{\infty} \frac{(\gamma)_n (2n+\gamma)}{(\beta+1)_n} R_n^{(\alpha,\beta)}(x) \quad (x \in X_r). \tag{14}$$

Proof. Our starting point is the generating function given in (5).

$$(1+t)^{-\gamma} H[4xt/(1+t)^2] = \sum_{n=0}^{\infty} \frac{(\gamma)_n t^n}{(\beta+1)_n} R_n^{(\alpha,\beta)}(x) \quad (|t| < 1), \tag{15}$$

$$H(z) = {}_2F_1\left(\left.\begin{matrix} \frac{\gamma}{2}, \ \frac{\gamma+1}{2} \\ \beta+1 \end{matrix}\right| z\right). \tag{16}$$

Since $H(z)$ is analytic for $|z| < 1$, the results for $r = 1, 2$ follow immediately from Theorem 1 and the asymptotic estimates for $R_n^{(\alpha,\beta)}(x)$ given in (6),

$$\left.\begin{aligned} R_n^{(\alpha,\beta)}(x) &= A(\theta)\, n^{-\frac{1}{2}} \cos\{(n+\gamma/2)\,\theta - \tfrac{1}{2}\pi\,(\tfrac{1}{2}+\beta)\}\,[1+O(n^{-1})],\\ n &\to \infty, \quad x = (1-\cos\theta)/2, \quad 0 < x < 1. \end{aligned}\right\} \tag{17}$$

A is a bounded function of θ independent of n. (We leave it to the reader to verify that not only (15) but any generating function of the form $\Psi(t)\, G\,[4xt/(1+t)^2]$, where Ψ, G are analytic in appropriate regions, satisfies the conditions of Theorem 1.)

When $r = 3$, more than Theorem 1 is needed, since $G(1, t)$ is not analytic for $t = 1$. Note, however, that if $\gamma < 0$ and $\alpha < -1$ the convergence of (14) for all $x \in X_3$ may be inferred from (17) and results in (7).

Let, then, $x = 1$ and put

$$L(t) = (1+t)^{-\gamma} H[4t/(1+t)^2], \tag{18}$$

$$\left(t\frac{d}{dt}+\frac{\gamma}{2}\right) L\,(t) = \frac{\gamma(\gamma+1)\,t(1-t)}{(1+t)\,(\beta+1)} L^*(t) + \frac{\gamma(1-t)}{2(1+t)} L(t) \quad (|t| < 1), \tag{19}$$

where L^* is L with α replaced by $\alpha+1$ and β by $\beta+1$.

Now, the behaviour of $L(t)$ near $t = 1$ is known; see, for example ((8), eq. 2·10). We have

$$L(t) = O[(1-t)^{2\alpha} \ln(1-t)] + O(1) \tag{20}$$

$$L^*(t) = O[(1-t)^{-2\alpha-2} \ln(1-t)] + O(1) \tag{21}$$

$$|t| \to 1, \quad |\arg(1-t)| < \pi.$$

Thus the hypotheses of the theorem guarantee that

$$\lim_{t\to 1-} \left(t\frac{d}{dt}+\frac{\gamma}{2}\right) L(t) = 0. \tag{22}$$

Consequently, Abel's theorem applies, and gives the result for X_3.

The third case is rather interesting, since the polynomials $R_n^{(\alpha,\beta)}(x)$ are orthogonal over the interval $[0, 1]$ (the weight function being $(1-x)^\alpha x^\beta$), the series sums to zero for $0 < x < 1$, and yet its coefficients are not all zero. Of course, the argument of section 1 does not apply here, since the series does not converge boundedly for all $0 \leqslant x \leqslant 1$.

For those values of x, α, β for which the convergence is absolute, (14) follows by substitution of the identity given in (9)

$$(2n+\gamma)\, R_n^{(\alpha,\beta)}(x) = (n+\gamma)\, R_n^{(\alpha+1,\beta)}(x) - (n+\beta)\, R_{n-1}^{(\alpha+1,\beta)}(x) \quad (n \geqslant 1), \tag{23}$$

and rearranging the terms.

A result more general than (14) which applies to series of the hypergeometric polynomials

$${}_{P+2}F_{P+1}\left(\begin{matrix} -n, n+\gamma, a_1, a_2, \ldots, a_P \\ b_1, b_2, \ldots, b_{P+1} \end{matrix}\middle| x\right) \tag{24}$$

can be demonstrated by using Theorem 1 on a generating function given in (10).

Kogbetliantz (11) has proved that, if the ultraspherical series

$$\sum_{n=0}^{\infty} c_n R_n^{(\alpha,\alpha)}(x) \tag{25}$$

converges with the sum zero everywhere in $[0, 1]$ (except, perhaps, at 0 and 1 and on a set of interior points of measure zero, where it may diverge, or converge with a sum different from 0) then $c_n = 0$ for all n. Used in his proof, however (but nowhere explicitly stated), is the hypothesis that $\alpha \geqslant -\frac{1}{2}$, see (11), p. 167. The same author has discussed at length the Cesàro summability of the series (14) when $\alpha = \beta$, see (12).

We now turn to an application of the above theorem.

Although uniqueness theorems for linear elliptic partial differential equations defined in a bounded domain D with coefficients continuous in $\bar{D}$ have been known for some time (13) it is only recently that uniqueness theorems have been derived for equations whose coefficients have singularities in the domain in question (14). Here we consider the singular partial differential equation

$$L_\nu(u) \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\nu}{y}\frac{\partial u}{\partial y} - \lambda^2 u = 0, \tag{26}$$

in a bounded domain D whose intersection with the x-axis is an open interval, and where ν is a real number, $\lambda > 0$. We shall now establish a uniqueness theorem for this equation and use Theorem 2 to explore its limitations.

In what follows, let ∂D denote the boundary of D.

THEOREM 3. *Let $\nu \geqslant -\frac{1}{2}$ and $g(x, y) \in C^0(\partial D)$. Then there is at most one solution $u(x, y)$ of $L_\nu(u) = 0$ such that $u(x, y) \in C^2(D) \cap C^0(\bar{D})$, $u(x, y) = u(x, -y)$ and $u(x, y) = g(x, y)$ on ∂D.*

This result is the best possible in the following sense: if $\nu < -\frac{1}{2}$, $2\nu \neq -1, -2, \ldots$, there are domains where, if any solution at all of $L_\nu(u) = 0$ exists satisfying the stated conditions, then that solution is not unique.

Proof. Assume $u(x, y)$ satisfies the conditions of the theorem. If u achieves its positive maximum in D and not on ∂D, this point must be on the x-axis, by the Hopf maximum principle (13). The fact that u is even in y implies that

$$\left.\frac{1}{y}\frac{\partial u}{\partial y}\right|_{y=0} = \left.\frac{\partial^2 u}{\partial y^2}\right|_{y=0}, \tag{27}$$

and so
$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_0,0)} + (1+2\nu)\left.\frac{\partial^2 u}{\partial y^2}\right|_{(x_0,0)} - \lambda^2 u(x_0, 0) = 0, \tag{28}$$

and if $(x_0, 0)$ is this maximum point,

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_{(x_0,0)} \leqslant 0, \quad \left.\frac{\partial^2 u}{\partial y^2}\right|_{(x_0,0)} \leqslant 0, \tag{29}$$

By hypothesis, $(1+2\nu) \geqslant 0$, $\lambda^2 u(x_0, 0) > 0$. But this makes (28) absurd so $u(x, y)$ cannot achieve its positive maximum in D. By replacing $u(x, y)$ by $-u(x,y)$, one finds similarly that $u(x, y)$ cannot achieve its negative minimum in D. If two solutions of (26) are equal to $g(x, y)$ on ∂D, then their difference satisfies (26) and vanishes on ∂D. But such a solution, not to be identically zero, must possess a positive maximum or a negative minimum in D. This is impossible, and the first part of the theorem is established.

We now use Theorem 2 to prove the last part of the theorem. The domain whose existence is asserted we will take to be the unit disk, Ω.

Consider the Bessel–Gegenbauer series

$$w(x,y) = r^{-\nu} \sum_{n=0}^{\infty} \frac{(n+\nu)\, I_{\nu+n}(\lambda r)}{I_{\nu+n}(\lambda)} C_n^{\nu}(\cos\theta), \tag{30}$$

vhere $I_{\nu+n}$ is the modified Bessel function of the first kind. Assume $\nu < -\frac{1}{2}$, $2\nu \neq -1, -2, \ldots$. From the series representation of $I_{\nu+n}$ we conclude that

$$I_{\nu+n}(z) = \frac{(z/2)^{n+\nu}}{\Gamma(n+\nu+1)}[1+O[(n+\nu)^{-1}]] \quad (n \to \infty), \tag{31}$$

ınd hence the differential operator L_ν can be applied termwise to the series (30) for $r < 1$. Since each term of the series satisfies (26), we infer that $L_\nu(w) = 0$ in Ω. Using (31) we can write (30) as

$$w(x,y) = \sum_{n=0}^{\infty} (n+\nu)\, r^n \left\{1 + \frac{M_n(\nu,r)}{(n+\nu)}\right\} C_n^{\nu}(\cos\theta), \tag{32}$$

vhere $M_n(\nu, r)$ is a bounded function of n for $(x,y) \in \overline{\Omega}$.

Also, $|C_n(\cos\theta)| = O(n^{\nu-1})$ uniformly for $\theta \epsilon [0,\pi]$, see (15), and so for $\nu < 0$, the series

$$\sum_{n=0}^{\infty} r^n M_n(\nu,r)\, C_n^{\nu}(\cos\theta) \tag{33}$$

onverges uniformly in $\overline{\Omega}$ and hence defines a continuous function there. By using a :nown result ((12), eq. (7)) and Theorem 2, we find that

$$\nu(1-r^2)(1-2r\cos\theta+r^2)^{-\nu-1} = \sum_{n=0}^{\infty} (n+\nu)\, r^n\, C_n^{\nu}(\cos\theta) \tag{34}$$

ı $\overline{\Omega}$. Hence if $\nu < -\frac{1}{2}$ the series on the right-hand side of (32) defines a continuous unction in $\overline{\Omega}$ so $w(x,y) \in C^0(\overline{\Omega})$. From (31) and the previously mentioned bound on the łegenbauer polynomials we infer that $w(x,y) \in C^2(\Omega)$. Obviously, $w(x,y) = w(x,-y)$. 'rom Theorem 2 with $\alpha = \beta = \nu - \frac{1}{2}$ and x replaced by $\frac{1}{2}(1+\cos\theta)$ we can conclude that $w(x,y) = 0$ on $\partial\Omega$. Hence $w(x,y)$ satisfies the conditions of the theorem but is not lentically zero. (To show this, let r tend to zero in (30).) The final statement of the heorem now follows.

For $\lambda = 0$, Theorem 3 was proved by Parter (14). In his work the function correspond- ıg to our $w(x,y)$ was constructed from a generating function for Gegenbauer poly- omials. This method fails in the case of the equation (26), since no generating function s known which satisfies the equation.

[*Added in proof*]: Prof. Richard Askey has kindly pointed out to us that the Łogbetliantz theorem referred to above is false, and that the problem of haracterizing uniqueness sets is still unsolved even for Fourier series (the case $\alpha = \beta = -\frac{1}{2}$).

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ON THE FACTORIZATION OF A CLASS OF DIFFERENCE OPERATORS[1]

BY JET WIMP AND JERRY FIELDS

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The differential equation for the Meijer G-function (generalized hypergeometric function) with respect to the argument z, [1], can be written in an elegant factored form using the differential operator $z(d/dz)$. Recently, [2], [3], it has been found that particular Meijer G-functions satisfy difference equations with respect to a parameter, and it is the purpose of this paper to deduce analogous factored forms for these difference equations.

Consider the function

$$G(x) = \frac{1}{2\pi i}\int_L z^s\Omega(s)K(s,x,y)ds, \tag{1}$$

$$\Omega(s) = \frac{\Gamma(c-s)\prod_{j=1}^{m}\Gamma(b_j-s)\Gamma(1-c+s)\prod_{j=1}^{k}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=k+1}^{p}\Gamma(a_j-s)}, \tag{2}$$

$$0 \leqq m \leqq q, \quad 0 \leqq k \leqq p; \qquad a_j \neq b_i, \quad 1 \leqq j \leqq k, \quad 1 \leqq i \leqq m,$$

$$K(s,x,y) = \Gamma(x+\delta s)/\Gamma(x+y+\epsilon s), \quad \epsilon \text{ and } \delta \text{ integers}, \ \delta \geqq 0, \tag{3}$$

where L is an infinite loop contour which separates the poles of $\Gamma(x+\delta s)\cdot\Gamma(1-c+s)\prod_{j=1}^{k}\Gamma(1-a_j+s)$ from those of $\Gamma(c-s)\prod_{j=1}^{m}\Gamma(b_j-s)$. Here and in what follows, we tacitly assume that the complex quan-

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tities a_i, b_j, c, x, y and z are such that the contour L actually exists. For more details about such integrals, see [1, p. 20].

We define two linear difference operators with respect to x,

$$\mathfrak{A}(\mu, x, y) = \alpha\mathfrak{J} + \beta\mathfrak{E}, \qquad \alpha = (x - \mu\delta)/\Delta, \quad \beta = (\epsilon\mu - x - y)/\Delta,$$

$$\text{(4)} \quad \mathfrak{A}^*(x, y) = \lim_{\mu\to\infty} \frac{\mathfrak{A}(\mu, x, y)}{\mu} = \alpha^*\mathfrak{J} + \beta^*\mathfrak{E},$$

$$\alpha^* = -\delta/\Delta, \quad \beta^* = \epsilon/\Delta, \quad \Delta = x(\epsilon - \delta) - y\delta \neq 0$$

where $\mathfrak{E}$ is the shift operator $\mathfrak{E}f(x) = f(x+1)$, and $\mathfrak{J}$ is the identity operator. Direct computation shows that

$$\text{(5)} \quad \begin{aligned} \mathfrak{A}(\mu, x, y)K(s, x, y) &= K(s, x, y + 1)(\mu + s), \\ \mathfrak{A}^*(x, y)K(s, x, y) &= K(s, x, y + 1). \end{aligned}$$

Finally, we set

$$\text{(6)} \quad \begin{aligned} \mathfrak{B} = z\mathfrak{E}^\delta &\prod_{j=1}^{p} \mathfrak{A}(1 - a_j, x, y + u + p - j) \prod_{j=1}^{u} \mathfrak{A}^*(x, y + u - j) \\ &+ (-1)^{m+p+k} \prod_{j=1}^{q} \mathfrak{A}(-b_j, x, y + v + q - j) \prod_{j=1}^{v} \mathfrak{A}^*(x, y + v - j), \\ u = \max\,&[0, q - p + \epsilon - \delta], \qquad v = \max\,[0, p - q + \delta - \epsilon]. \end{aligned}$$

In the ordinary product notation used above, the order of the factors must be interpreted as follows:

$$\prod_{j=1}^{r} P_j = P_1P_2 \cdots P_r.$$

Our principal result is the following

THEOREM. *For the a_i, b_j, c, x, y and z as previously restricted,*

$$\text{(7)} \quad \mathfrak{B}G(x) = (-1)^{p+k} \frac{z^c\Gamma(x + \delta c)}{\Gamma(x + y + v + q + \epsilon c)} \cdot \frac{\prod_{j=1}^{k} \Gamma(1 + c - a_j) \prod_{j=1}^{m} \Gamma(1 + b_j - c)}{\prod_{j=m+1}^{q} \Gamma(c - b_j) \prod_{j=k+1}^{p} \Gamma(a_j - c)}.$$

PROOF. By applying $\mathfrak{B}$ directly to the integrand of (1), and using (5), together with

$$\Omega(s+1) = \Omega(s)(-1)^{m+k+p+1} \prod_{j=1}^{p} (1-a_j+s) \Big/ \prod_{j=1}^{q} (1-b_j+s), \tag{8}$$

one readily verifies that

$$\begin{aligned} \mathfrak{B}G(x) &= \frac{1}{2\pi i}\int_L z^{s+1}\Omega(s) \prod_{j=1}^{p} (1-a_j+s) K(s, x+\delta, y+u+p)ds \\ &\quad - \frac{1}{2\pi i}\int_{L-1} z^{s+1}\Omega(s) \prod_{j=1}^{p} (1-a_j+s) K(s+1, x, y+v+q)ds. \end{aligned} \tag{9}$$

As $K(s, x+\delta, y+u+p) = K(s+1, x, y+u+p+\delta-\epsilon)$, and $u+p+\delta-\epsilon = v+q$, $\mathfrak{B}G(x)$ is just equal to the sum of the residues of $z^{s+1}\Omega(s) \cdot \prod_{j=1}^{p}(1-a_j+s)K(s+1, x, y+v+q)$ contained in the region between L and $L-1$. By inspection, we see the only possible residue is at $s=c-1$, and (9) reduces to (7).

REMARK 1. It should be noted that there is a certain arbitrariness in the definition of $\mathfrak{B}$, which is attributable to the symmetry property

$$\mathfrak{A}(\mu_2, x, y+1)\mathfrak{A}(\mu_1, x, y) = \mathfrak{A}(\mu_1, x, y+1)\mathfrak{A}(\mu_2, x, y). \tag{10}$$

Clearly, $\mathfrak{B}$ can be rewritten in the form

$$\begin{aligned} \mathfrak{B} &= \sum_{j=0}^{\tau} [A_j + zB_j]\mathfrak{E}^j, \qquad B_0 = 0, \\ \tau &= \max\{q, q+\epsilon, p+\delta, p+\delta-\epsilon\}. \end{aligned} \tag{11}$$

REMARK 2. In reference [3] it was shown that the extended Jacobi functions

$$\begin{aligned} &{}_{r+3}F_t\left(\begin{matrix} -n, n+\lambda, \sigma_r, 1 \\ \rho_t \end{matrix}\middle| z\right) \\ &= \frac{\Gamma(n+1)}{\Gamma(n+\lambda)} \frac{\prod_{j=1}^{t}\Gamma(\rho_j)}{\prod_{j=1}^{r}\Gamma(\sigma_j)} G^{1,r+2}_{r+3,t+1}\left(z\middle| \begin{matrix} 1-n-\lambda, 1-\sigma_r, 0, n+1 \\ 0, 1-\rho_t \end{matrix}\right) \end{aligned} \tag{12}$$

and the extended Laguerre functions

$$ \text{(13)} \quad {}_{r+2}F_t\left(\begin{matrix}-n, \sigma_r, 1\\ \rho_t\end{matrix}\middle| z\right) = \frac{\Gamma(n+1)\prod_{j=1}^{t}\Gamma(\rho_j)}{\prod_{j=1}^{r}\Gamma(\sigma_j)} G_{r+2,t+1}^{1,r+1}\left(z\middle|\begin{matrix}1-\sigma_r, 0, n+1\\ 0, 1-\rho_t\end{matrix}\right) $$

satisfy normalized difference equations involving a difference operator of the form (11) with

$$ \text{(14)} \qquad \tau = \max[r+2, t] $$

and

$$ \text{(15)} \qquad \tau = \max[r+1, t], $$

respectively. Furthermore, it was shown that these functions satisfied no other difference equation so normalized of orders $\leqq$ those given by (14) and (15), respectively, provided certain conditions on ρ_i, σ_j, λ were satisfied.

But the G-function on the right in (12) is the integral (1) with

$$ \text{(16)} \quad \begin{matrix} m=0, & k=p=r, & q=t, & c=0, & x=n+\lambda, \\ & y=1-\lambda, & \delta=1, & \epsilon=-1, & \end{matrix} $$

while the right-hand side of (13) is, apart from a constant multiple, (1) with

$$ \text{(17)} \quad \begin{matrix} m=0, & k=p=r, & q=t, & c=0, & x=n+1, \\ & y=0, & \delta=0, & \epsilon=-1. & \end{matrix} $$

Furthermore, the formula for τ in (11) gives (14) for the values (16), and (15) for the values (17). In view of the aforementioned uniqueness of the difference equations, it follows that (6) will yield a factorization of those difference equations given in [**3**].

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Recursion Formulae for Generalized Hypergeometric Functions[1]

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I. INTRODUCTION AND NOTATION

In Luke [*1*] and Fields [*2*], rational approximations to certain classes of hypergoemetric functions are developed. The results include as special cases the main and off diagonal entries of the Padé matrix [*3*, *4*] for the Gaussian hypergeometric function, one of whose numerator parameters is unity. A well-known property of this matrix is that the numerator and denominator of each entry satisfy the same three-term recurrence formula. Recently, Wimp [*5*] derived explicit recursion formulae for a certain class of hypergeometric functions closely related to the denominator polynomials of the Luke and Fields approximations. Thus, it is natural to ask, using a modified form of Wimp's analysis, whether the Luke and Fields approximations satisfy recurrence properties similar to those of the Padé matrix. This and related questions are answered in this paper.

The generalized hypergeometric function [*6*] is defined by the formal expression

$$ {}_pF_q(z) = {}_pF_q\left(\begin{matrix}\alpha_1, \ldots, \alpha_p \\ \beta_1, \ldots, \beta_q\end{matrix}\middle| z\right) = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p} (\alpha_j)_k}{\prod_{j=1}^{q} (\beta_j)_k} \cdot \frac{z^k}{k!}, \tag{1.1} $$

where

$$ (\sigma)_\mu = \frac{\Gamma(\sigma+\mu)}{\Gamma(\sigma)}. $$

We assume that no β_j is a nonpositive integer. For ease in writing, we employ the contracted notation

$$ {}_pF_q(z) = {}_pF_q\left(\begin{matrix}\alpha_p \\ \beta_q\end{matrix}\middle| z\right) = \sum_{k=0}^{\infty} \frac{(\alpha_p)_k}{(\beta_q)_k} \cdot \frac{z^k}{k!}. \tag{1.2} $$

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Thus $(\alpha_p)_k$ is to be interpreted as $\prod_{j=1}^{p}(\alpha_j)_k$ and similarly for $(\beta_q)_k$. Similar notations such as $\Gamma(\alpha_p)$ standing for $\prod_{j=1}^{p}\Gamma(\alpha_j)$, and $(\alpha_p)^*_{-\alpha_l}$ standing for

$$\prod_{\substack{j=1\\ j\neq l}}^{p}(\alpha_j)_{-\alpha_l}$$

will be used throughout this paper. Considered as a power series in z, ${}_pF_q(z)$ has a radius of convergence equal to infinity if $p \leqslant q$, unity if $p = q+1$, and (in general) zero if $p \geqslant q+2$. If one of the α_j is a negative integer, the infinite series in (1.1) terminates. If no α_j is a negative integer, a meaning can still be given to ${}_pF_q(z)$, $p \geqslant q+2$, by considering it as the asymptotic expansion as $z \to 0$, of a certain type of contour integral.

More generally, we define Meijer's G-function [6] by

$$G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_p\\ \\ b_q\end{matrix}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{n}\Gamma(1-a_j+s)}{\prod_{j=m+1}^{q}\Gamma(1-b_j+s)\prod_{j=n+1}^{p}\Gamma(a_j - s)} z^s\,ds, \tag{1.3}$$

where an empty product is interpreted as 1, $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$, the parameters are such that no pole of $\Gamma(b_j - s), j = 1, \ldots, m$ coincides with any pole of $\Gamma(1 - a_k + s)$, $k = 1, \ldots, n$, and where the path L runs parallel to the imaginary axis, and is indented to separate the poles of $\Gamma(b_m - s)$ from the poles of $\Gamma(1 - a_n + s)$. The above integral is well defined if $p + q < 2(m+n)$ and $|\arg z| < [(m+n) - (p+q)/2]\pi$. If all the poles of the integrand in (1.3) are simple, it is easy to see from the residue theorem, that

$$G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_p\\ \\ b_q\end{matrix}\right)$$

can be represented as a sum of well-defined hypergeometric functions, e.g.

$$G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_p\\ \\ b_q\end{matrix}\right)$$

$$= \sum_{h=1}^{m} \frac{\prod_{\substack{j=1\\ j\neq h}}^{m}\Gamma(b_j - b_h)\prod_{j=1}^{n}\Gamma(1+b_h-a_j)\,z^{b_h}}{\prod_{j=m+1}^{q}\Gamma(1+b_h-b_j)\prod_{j=n+1}^{p}\Gamma(a_j - b_h)}\,{}_{p+1}F_q\left(\begin{matrix}1, 1+b_h-a_p\\ \\ 1+b_h-b_q\end{matrix}\,\middle|\,(-1)^{p-m-n}z\right),$$

$$p < q \quad \text{or} \quad p = q \quad \text{and} \quad |z| < 1. \tag{1.4}$$

A similar expansion holds if $p>q$ or $p=q$ and $|z|>1$, and follows directly from (1.4) and the functional relationship

$$G^{m,n}_{p,q}\left(z^{-1}\left|\begin{matrix}a_p\\ \\ b_q\end{matrix}\right.\right)=G^{n,m}_{q,p}\left(z\left|\begin{matrix}1-b_q\\ \\ 1-a_p\end{matrix}\right.\right). \tag{1.5}$$

Both functional relationships (1.5) and

$$z^c\,G^{m,n}_{p,q}\left(z\left|\begin{matrix}a_p\\ \\ b_q\end{matrix}\right.\right)=G^{m,n}_{p,q}\left(z\left|\begin{matrix}c+a_p\\ \\ c+b_q\end{matrix}\right.\right), \tag{1.6}$$

follow directly from the integral definition (1.3).

A special case of (1.4) is

$$E_{p,q}(z)\equiv G^{1,p}_{p,q+1}\left(-z\left|\begin{matrix}1-\alpha_p\\ \\ 0,1-\beta_q\end{matrix}\right.\right)$$

$$=\frac{\Gamma(\alpha_p)}{\Gamma(\beta_q)}\,{}_pF_q\left(\begin{matrix}\alpha_p\\ \\ \beta_q\end{matrix}\left|z\right.\right);\qquad p<q+1\quad\text{or}\quad p=q+1,\quad |\arg(-z)|<\pi. \tag{1.7}$$

Thus, $E_{q+1,q}(z)$ analytically extends ${}_{q+1}F_q(z)$ into the region $|\arg(1-z)|<\pi$. Moreover, it can be shown [7] that for $p\geqslant q+2$,

$$E_{p,q}(z)\sim\frac{\Gamma(\alpha_p)}{\Gamma(\beta_q)}\,{}_pF_q\left(\begin{matrix}\alpha_p\\ \\ \beta_q\end{matrix}\left|z\right.\right),$$

$$|\arg(-z)|<(p+1-q)\pi/2,\qquad z\to 0. \tag{1.8}$$

The formal Luke and Fields rational approximations to $E_{p,q}(z)$, $\psi_n(z,\gamma)/f_n(\gamma)$, are defined as follows. For $a=0$ or 1, and the parameters $A_{n,k}$, γ arbitrary, set

$$f_n^{[r]}(\gamma)=\sum_{k=r}^{n}A_{n,k}\gamma^k;\qquad f_n(\gamma)=f_n^{[0]}(\gamma), \tag{1.9}$$

$$\psi_n(z,\gamma)=\sum_{k=0}^{n}A_{n,k}\gamma^kP_{k-a+1}(z),$$

$$=\sum_{r=0}^{n-a}\frac{\Gamma(r+\alpha_p)}{\Gamma(r+\beta_q)\,r!}z^rf_n^{[r+a]}(\gamma),$$

$$=\sum_{k=a}^{n}z^{-k}\sum_{r=k}^{n}\frac{A_{n,r}\Gamma(r-k+\alpha_p)}{\Gamma(r-k+\beta_q)(r-k)!}(\gamma z)^r, \tag{1.10}$$

$$P_k(z)=\sum_{j=0}^{k-1}\frac{\Gamma(j+\alpha_p)}{\Gamma(j+\beta_q)\,j!}z^j, \qquad P_0(z)=0. \tag{1.1}$$

It was shown in [2] that if

$$A_{n,k}=\frac{(-n)_k\,(n+\lambda)_k\,(\beta_q-a)_k}{(\beta+1)_k\,(\alpha_p+1-a)_k}; \qquad \lambda,\beta\,\text{arbitrary}, \tag{1.1}$$

then

$$E_{p,q}(z)=\lim_{n\to+\infty}\frac{\psi_n(z,\gamma)}{f_n(\gamma)}, \tag{1.1}$$

under quite general restrictions on p, q, z, γ, etc. The significance of the par meter a is plainly seen, if in the last line of (1.10) one successively sets $z=\gamma$ and $\gamma=0$. For then $\psi_n(\infty,0)$ equals zero if $a=1$, and is not equal to zero $a=0$. Classically, the cases $a=0$ and $a=1$ correspond to taking the odd ar even convergents, respectively, of certain continued fractions, see [6, 8].

We note that if $A_{n,k}$ is chosen as in (1.12), the denominator polynomi $f_n(\gamma)$ is of the general hypergeometric form

$$_{r+2}F_s\left(\begin{matrix}-n,n+\lambda,\alpha_r\\ \\ \beta_s\end{matrix}\middle| z\right) \tag{1.1}$$

which is known as the extended Jacobi polynomial. A limiting form of t extended Jacobi polynomial is the extended Laguerre polynomial

$$_{r+1}F_s\left(\begin{matrix}-n,\alpha_r\\ \\ \beta_s\end{matrix}\middle| z\right). \tag{1.1}$$

If n is not an integer, (1.14) and (1.15) are known as extended Jacobi ar Laguerre functions, respectively. In Section II, explicit linear recursi equations for such polynomials (functions) are derived. In Section III, line recursion equations for the corresponding numerator polynomials, $\psi_n(z,\gamma)$, a also derived. Our results are stated quite generally.

In Section IV, we relate the material of the previous sections to the proble of finding recursion relationships for the coefficients in the expansion Meijer G-functions in series of extended Jacobi and Laguerre polynomia In particular, it is shown in [9, 10, 11], that under sufficient restrictions,

$$G^{m,k+r}_{p+r,q+s}\left(\omega z\middle|\begin{matrix}c_r,a_p\\ \\ b_q,d_s\end{matrix}\right)=\frac{\Gamma(1-c_r)}{\Gamma(1-d_s)}\sum_{n=0}^{\infty}\frac{(-)^n(2n+\lambda)\,\Gamma(n+\lambda)}{n!}$$

$$\times\, G^{m,k+1}_{p+1,q+2}\left(\omega\middle|\begin{matrix}0,a_p\\ \\ b_q,n,-n-\lambda\end{matrix}\right)\,{}_{r+2}F_s\left(\begin{matrix}-n,n+\lambda,1-c_r\\ \\ 1-d_s\end{matrix}\middle| z\right), \tag{1.1}$$

$$G_{p+r,\,q+s}^{m,\,k+r}\left(\omega z\,\middle|\,\begin{matrix}c_r,\, a_p\\ \\ b_q,\, d_s\end{matrix}\right)=\frac{\Gamma(1-c_r)}{\Gamma(1-d_s)}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}$$
$$\times\, G_{p+1,\,q+1}^{m,\,k+1}\left(\omega\,\middle|\,\begin{matrix}0,\, a_p\\ \\ b_q,\, n\end{matrix}\right){}_{r+1}F_s\left(\begin{matrix}-n,\, 1-c_r\\ \\ 1-d_s\end{matrix}\,\middle|\, z\right). \qquad (1.17)$$

For generalized hypergeometric functions, (1.16) and (1.17) can be interpreted as

$${}_{p+r}F_{q+s}\left(\begin{matrix}\alpha_p,\,\sigma_r\\ \\ \beta_q,\,\rho_s\end{matrix}\,\middle|\,\omega z\right)=\sum_{n=0}^{\infty}\frac{(\alpha_p)_n\,(-\omega)^n}{(\beta_q)_n(n+\lambda)_n\, n!}$$
$$\times\,{}_pF_{q+1}\left(\begin{matrix}n+\alpha_p\\ \\ n+\beta_q,\, 2n+\lambda+1\end{matrix}\,\middle|\,\omega\right){}_{r+2}F_s\left(\begin{matrix}-n,\, n+\lambda,\,\sigma_r\\ \\ \rho_s\end{matrix}\,\middle|\, z\right), \qquad (1.18)$$

and

$${}_{p+r}F_{q+s}\left(\begin{matrix}\alpha_p,\,\sigma_r\\ \\ \beta_q,\,\rho_s\end{matrix}\,\middle|\,\omega z\right)=\sum_{n=0}^{\infty}\frac{(\alpha_p)_n(-\omega)^n}{(\beta_q)_n\, n!}$$
$$\times\,{}_pF_q\left(\begin{matrix}n+\alpha_p\\ \\ n+\beta_q\end{matrix}\,\middle|\,\omega\right){}_{r+1}F_s\left(\begin{matrix}-n,\,\sigma_r\\ \\ \rho_s\end{matrix}\,\middle|\, z\right). \qquad (1.19)$$

II. Recursion Formulae for the Extended Jacobi and Laguerre Functions

In the following, we shall derive a linear, nonhomogeneous difference equation for the generalized Jacobi function,

$$\mathscr{E}_n(z,\lambda)={}_{r+3}F_s\left(\begin{matrix}-n,\, n+\lambda,\,\alpha_r,\, 1\\ \\ \beta_s\end{matrix}\,\middle|\, z\right);\qquad n\text{ arbitrary},$$
$$r+3\leqslant s,\quad\text{or}\quad r+2=s\quad\text{and}\quad|\arg(1-z)|<\pi. \qquad (2.1)$$

Complementary to $\mathscr{E}_n(z,\lambda)$ is the function

$$\mathscr{K}_n(z,\lambda)=\frac{(\beta_s-1)\, z^{-1}}{(n+1)(n+\lambda-1)(\alpha_r-1)}$$
$$\times\,{}_{s+1}F_{r+2}\left(\begin{matrix}2-\beta_s,\, 1\\ \\ 2+n,\, 2-n-\lambda,\, 2-\alpha_r\end{matrix}\,\middle|\,\frac{(-1)^{s-r}}{z}\right),$$
$$r+1\geqslant s,\quad\text{or}\quad r+2=s\quad\text{and}\quad|\arg(1-1/z)|<\pi, \qquad (2.2)$$

in the sense that both are particular solutions of the differential equation

$$[(\delta+\beta_s-1)-z(\delta-n)(\delta+n+\lambda)(\delta+\alpha_r)]\,Y(z)=(\beta_s-1), \qquad (2.3$$

where

$$(\delta+\alpha_r)=\prod_{j=1}^{r}(\delta+\alpha_j), \text{ etc.}, \qquad \delta=z\frac{d}{dz}. \qquad (2.4$$

We shall not only show that $\mathscr{E}_n(z,\lambda)$ and $\mathscr{K}_n(z,\lambda)$ satisfy the same linear, non-homogeneous difference equation, but that a properly normalized basis of th- related, homogeneous differential equation

$$[(\delta+\beta_s-1)-z(\delta-n)(\delta+n+\lambda)(\delta+\alpha_r)]\,Y(z)=0, \qquad (2.5$$

also satisfies the related, homogeneous difference equation.

To describe these bases, normalized with respect to n, it is convenient t- write down the following sets of conditions:

$$\left.\begin{array}{l} r+3\leqslant s, \quad \text{or} \quad r+2=s \quad \text{and} \quad |\arg(1-z)|<\pi, \\ \text{no two of the parameters, } \beta_h (h=1,\ldots,s), \text{ differ by an} \\ \text{integer,} \end{array}\right\} C_{0,\lambda}$$

$$\left.\begin{array}{l} r+1\geqslant s, \quad \text{or} \quad r+2=s \quad \text{and} \quad |\arg(1-1/z)|<\pi, \\ \text{no two of the parameters, } -n, n+\lambda, \alpha_k (k=1,\ldots,r), \text{ differ} \\ \text{by an integer.} \end{array}\right\} C_{\infty,\lambda}$$

Under condition $C_{0,\lambda}$, we take for our normalized basis,

$$\begin{aligned} &\mathscr{F}_{n,h}(z,\lambda) \\ &= (n+\beta_h)_{1-\beta_h}(n+\lambda)_{1-\beta_h}\, z^{1-\beta_h} \\ &\quad \times {}_{r+3}F_s\left(\begin{matrix}1, 1-\beta_h-n, 1-\beta_h+n+\lambda, 1-\beta_h+\alpha_r \\ 1-\beta_h+\beta_s\end{matrix}\middle|\, z\right) \qquad (2.6 \\ &\qquad\qquad h=1,\ldots,s. \end{aligned}$$

Alternatively, under condition $C_{\infty,\lambda}$, we take for our normalized basis,

$$\begin{aligned} &\mathscr{G}_{n,k}(z,\lambda) \\ &= (n+1+\alpha_k)_{-\alpha_k}(n+\lambda)_{-\alpha_k}\, z^{-\alpha_k} \\ &\quad \times {}_{s+1}F_{r+2}\left(\begin{matrix}1, 1+\alpha_k-\beta_s \\ 1+\alpha_k+n, 1+\alpha_k-n-\lambda, 1+\alpha_k-\alpha_r\end{matrix}\middle|\, \frac{(-1)^{s-r}}{z}\right), \\ &\qquad\qquad k=1,\ldots,r. \qquad (2.7 \end{aligned}$$

$$\mathscr{G}_{n,r+1}(z,\lambda)$$

$$= \frac{\Gamma(n+1)\,\Gamma(2n+\lambda)\,\Gamma(n+\alpha_r)\,e^{in\phi}\,z^n}{\Gamma(n+\lambda)\,\Gamma(n+\beta_s)} \times {}_sF_{r+1}\left(\begin{matrix} 1-\beta_s-n \\ 1-2n-\lambda, 1-\alpha_r-n \end{matrix}\middle| \frac{(-1)^{s-r}}{z}\right), \tag{2.8}$$

$$\mathscr{G}_{n,r+2}(z,\lambda)$$

$$= \frac{\Gamma(n+1)\,\Gamma(n+\lambda+1-\beta_s)\,e^{in\phi(s-r-1)}\,z^{-n-\lambda}}{\Gamma(n+\lambda)\,\Gamma(2n+\lambda+1)\,\Gamma(n+\lambda+1-\alpha_r)} \times {}_sF_{r+1}\left(\begin{matrix} n+\lambda+1-\beta_s \\ 2n+\lambda+1, n+\lambda+1-\alpha_r \end{matrix}\middle| \frac{(-1)^{s-r}}{z}\right), \tag{2.9}$$

here $e^{i\phi} = -1$.

With these definitions, we state

THEOREM 2.1. *The functions $\mathscr{E}_n(z,\gamma)$ and $\mathscr{K}_n(z,\gamma)$ under the conditions on r, s nd z in $C_{0,\lambda}$ and $C_{\infty,\lambda}$, respectively, satisfy the difference equation*

$$\Phi_n(z,\lambda) + \sum_{m=1}^{t} [A_m(n,\lambda) + zB_m(n,\lambda)]\,\Phi_{n-m}(z,\lambda) = \frac{(\beta_s-1)(n+\lambda)_n}{(n+\beta_s-1)(n+\lambda-t)_n},$$

$$t = \max(r+2, s), \quad B_t(n,\lambda) = 0 \tag{2.10}$$

here

$$A_m(n,\lambda)$$

$$= \frac{(n+1-m)_m(2n+\lambda-2m)_{2m}(n-m-1+\beta_s)}{m!\,(n+\lambda-m)_m(2n+\lambda-t-m)_m(n-1+\beta_s)} \times {}_{s+2}F_{s+1}\left(\begin{matrix} -m, 2n+\lambda-t-m, n-m+\beta_s \\ 2n+\lambda+1-2m, n-m-1+\beta_s \end{matrix}\middle| 1\right),$$

$$= \frac{(-)^s(n+1-m)_m(2n+\lambda-2m)(2n+\lambda-t+1)_{t-1}(n+\lambda-t+1-\beta_s)}{(t-m)!\,(n+\lambda-m)_m(2n+\lambda-t-m)_m(n-1+\beta_s)} \times {}_{s+2}F_{s+1}\left(\begin{matrix} -t+m, 2n+\lambda-t-m, n+\lambda-t+2-\beta_s \\ 2n+\lambda+1-t, n+\lambda-t+1-\beta_s \end{matrix}\middle| 1\right), \tag{2.11}$$

$$B_m(n,\lambda)$$

$$=\frac{(n+1-m)_m(2n+\lambda-2m)_{2m}(n-m+\alpha_r)}{(m-1)!\,(n+\lambda-m)_m(2n+\lambda-t-m+1)_{m-1}(n-1+\beta_s)}$$

$$\times\,{}_{r+2}F_{r+1}\left(\begin{matrix}1-m,2n+\lambda-t-m+1,n-m+1+\alpha_r\\ 2n+\lambda+1-2m,n-m+\alpha_r\end{matrix}\middle|1\right),$$

$$=\frac{(-)^r(n+1-m)_m(2n+\lambda-2m)(2n+\lambda-t+1)_{t-1}(n+\lambda-t+1-\alpha_r)}{(t-m-1)!\,(n+\lambda-m)_m(2n+\lambda-t-m+1)_{m-1}(n-1+\beta_s)}$$

$$\times\,{}_{r+2}F_{r+1}\left(\begin{matrix}-t+m+1,2n+\lambda-t-m+1,n+\lambda-t+2-\alpha_r\\ 2n+\lambda+1-t,n+\lambda-t+1-\alpha_r\end{matrix}\middle|1\right).$$

(2.1

In addition, the functions $\mathscr{F}_{n,h}(z,\lambda)$ $(h=1,\ldots,s)$ and $\mathscr{G}_{n,k}(z,\lambda)$ $(k=1,\ldots,r+$ under the conditions $C_{0,\lambda}$ and $C_{\infty,\lambda}$, respectively, satisfy the difference equati

$$\Phi_n(z,\lambda)+\sum_{m=1}^{t}[A_m(n,\lambda)+zB_m(n,\lambda)]\Phi_{n-m}(z,\lambda)=0. \qquad (2.1$$

Finally, if no α_k is equal to any β_h, none of the above functions satisfy a nontrivi equation of the form specified of lower order than t.

Proof. By analytic continuation with respect to z, it is sufficient to prove t theorem with the conditions $C_{0,\lambda}$ and $C_{\infty,\lambda}$ strengthened to $|z|<1$ and $|z|>$ respectively. Tentatively, we assume that no α_k is equal to any β_h, and th we wish to prove

$$\sum_{m=0}^{t}[A_m(n,\lambda)+zB_m(n,\lambda)]\Phi_{n-m}(z,\lambda)=K(\Phi_n(z,\lambda)), \qquad A_0(n,\lambda)=1, \qquad (2.1$$

where $K(\Phi_n(z,\lambda))$ is a monomial in z, which depends upon the identity $\Phi_n(z,\lambda)$. A necessary and sufficient condition that $\mathscr{E}_n(z,\lambda)$, $\mathscr{K}_n(z,\lambda)$, $\mathscr{F}_{n,h}(z,$ or $\mathscr{G}_{n,k}(z,\lambda)$ satisfy (2.14), is that when these functions are substituted in (2.1 and the resulting equations are rearranged in powers of z, the coefficients z^j, z^{-j}, $z^{1-\beta_h+j}$ and $z^{1-\sigma_k-j}$ $(\sigma_k=\alpha_k$ $(k=1,\ldots,r)$, $\sigma_{r+1}=-n$, $\sigma_{r+2}=n+\lambda-$ respectively, are zero for $j=1,2,\ldots,$ while the terms corresponding to $j=$ reduce to $K(\Phi_n(z,\lambda))$. In order to state these conditions explicitly, we introdu the polynomials

$$X_t(w)=\sum_{m=0}^{t}\frac{(w-n)_m(w+n+\lambda-t)_{t-m}}{(-n)_m(n+\lambda-t)_{t-m}}A_m(n,\lambda),$$

$$Y_t(w)=-\sum_{m=0}^{t}\frac{(w-n-1)_m(w+n+\lambda-t-1)_{t-m}}{(-n)_m(n+\lambda-t)_{t-m}}B_m(n,\lambda). \qquad (2.1$$

hen, after some algebra, the above conditions can be written in the form

$$K(\mathscr{E}_n(z,\lambda)) = X_t(0),$$

$$K(\mathscr{K}_n(z,\lambda)) = -\frac{(\beta_s - 1)\; Y_t(0)}{(\alpha_r - 1)(n+1)(n+\lambda-t-1)},$$

$$K(\mathscr{F}_{n,h}(z,\lambda))$$
$$= (n+\beta_h)_{1-\beta_h}(n+\lambda-t)_{1-\beta_h} X_t(1-\beta_h)\, z^{1-\beta_h}, \qquad h = 1, \ldots, s,$$

$$K(\mathscr{G}_{n,k}(z,\lambda))$$
$$= -(n+1+\alpha_k)_{-\alpha_k}(n+\lambda-t)_{-\alpha_k}\, Y_t(1-\alpha_k)\, z^{1-\alpha_k}, \qquad k = 1, \ldots, r,$$

$$K(\mathscr{G}_{n,r+1}(z,\lambda))$$
$$= \frac{\Gamma(n+1)\,\Gamma(2n+\lambda-t)\,\Gamma(n+\alpha_r)\, e^{in\phi}}{\Gamma(n+\lambda-t)\,\Gamma(n+\beta_s)} - Y_t(1+n)\, z^{1+n},$$

$$K(\mathscr{G}_{n,r+2}(z,\lambda))$$
$$= \frac{-\Gamma(n+1)\,\Gamma(n+\lambda-t+1-\beta_s)\, e^{i\phi(s-r-1)(n-t)}}{\Gamma(n+\lambda-t)\,\Gamma(2n+\lambda-t+1)\,\Gamma(n+\lambda-t+1-\alpha_r)}$$
$$\times\; Y_t(1-n-\lambda+t)\, z^{1-n-\lambda+t}, \tag{2.16}$$

d

$$(w)(w-n-1)(w+n+\lambda-t-1)\prod_{k=1}^{r}(w+\alpha_k-1) = Y_t(w)\prod_{h=1}^{s}(w+\beta_h-1), \tag{2.17}$$

henever $w = j$, $-j$, $1-\beta_h+j$ or $1-\sigma_k-j$, for $\mathscr{E}_n(z,\lambda)$, $\mathscr{K}_n(z,\lambda)$, $\mathscr{F}_{n,h}(z,\lambda)$, $\mathscr{G}_{n,k}(z,\lambda)$, respectively.

As (2.17) can be viewed as a polynomial form in w of degree $\leqslant 2t$, (2.17) must tually hold for all w. Moreover, as the α_k and β_h are independent of n, and sumed unequal,

$$\prod_{h=1}^{s}(w+\beta_h-1)\left((w-n-1)(w+n+\lambda-t-1)\prod_{k=1}^{r}(w+\alpha_k-1)\right)$$

ust divide $X_t(w)$ $(Y_t(w))$, and the resulting polynomial will have degree $\max(r+2-s,0)$ $(\leqslant \max(0,s-r-2))$. Suppose $r+2 \leqslant s$. Then there ists a number C independent of w such that

$$X_t(w) = C\prod_{h=1}^{s}(w+\beta_h-1), \tag{2.18}$$

and substitution of this identity into (2.17) yields

$$Y_t(w) = C(w - n - 1)(w + n + \lambda - t - 1) \prod_{k=1}^{r} (w + \alpha_k - 1). \qquad (2.19)$$

The alternate assumption, $r + 2 \geqslant s$, again leads to (2.18) and (2.19). The valu of the constant C follows from $X_t(n)$, i.e.,

$$C = \frac{X_t(n)}{(n + \beta_s - 1)} = \frac{(2n + \lambda - t)_t\, A_0(n, \lambda)}{(n + \lambda - t)_t\,(n + \beta_s - 1)} = \frac{(n + \lambda)_n}{(n + \lambda - t)_n\,(n + \beta_s - 1)}. \qquad (2.20)$$

Moreover, it follows from (2.16), (2.18) and (2.19), that

$$\begin{aligned} K(\mathscr{E}_n(z, \lambda)) = K(\mathscr{H}_n(z, \lambda)) &= \frac{(\beta_s - 1)(n + \lambda)_n}{(n + \beta_s - 1)(n + \lambda - t)_n}, \\ K(\mathscr{F}_{n,h}(z, \lambda)) = K(\mathscr{G}_{n,k}(z, \lambda)) &= 0. \end{aligned} \qquad (2.21)$$

Finally, the values of $A_m(n,\lambda)$ ($B_m(n,\lambda)$) given in (2.11) ((2.12)) follow fror (2.18) ((2.19)) by an application of the following lemma.

LEMMA 2.1. *If $P_q(x)$ is a polynomial in x of degree q,*

$$P_q(x) = c \prod_{i=1}^{q} (x - \omega_i), \qquad (2.22)$$

and t is an integer $\geqslant q$, then $P_q(x)$ can be represented uniquely in the form

$$P_q(x) = \sum_{m=0}^{t} (x + \gamma)_m (x + \gamma + \epsilon)_{t-m}\, Q_m, \qquad (2.23)$$

$$\begin{aligned} Q_m &= \frac{(-)^q (t + \epsilon - 2m)(\gamma + \omega_q)\, c}{m!\,(\epsilon)_{t+1-m}} \\ &\quad \times {}_{q+2}F_{q+1}\left(\left.\begin{matrix} -m, m - \epsilon - t, 1 + \gamma + \omega_q \\ 1 - \epsilon, \gamma + \omega_q \end{matrix}\right| 1\right), \\ &= \frac{(-1)^{q+t}(\gamma + \epsilon + t - m + \omega_q)\, c}{(t - m)!\,(1 + \epsilon + t - 2m)_m} \\ &\quad \times {}_{q+2}F_{q+1}\left(\left.\begin{matrix} m - t, m - t - \epsilon, 1 + m - t - \gamma - \epsilon - \omega_q \\ 1 + 2m - t - \epsilon, m - t - \gamma - \epsilon - \omega_q \end{matrix}\right| 1\right), \end{aligned} \qquad (2.24)$$

provided $\epsilon \neq 0, \pm 1, \ldots, \pm(t - 1)$. Note that $Q_m = 0$ if $m \geqslant t + 1$. If $(\gamma + \omega_q)$ $(\gamma + \epsilon + \omega_q)$ are zero, limits must be taken in (2.24).

Proof. We first show that given $P_q(x)$, the coefficients Q_m, if they exist at all, are unique. Suppose there exists a set of $Q_m{}^*$'s which also satisfy (2.23). Then by subtraction

$$\sum_{m=0}^{t} (x+\gamma)_m(x+\gamma+\epsilon)_{t-m}(Q_m - Q_m{}^*) = 0 \tag{2.25}$$

for all x. In (2.25) put $x+\gamma=-j$, $j=0, 1, \ldots, t$. Then by Cramer's rule, $Q_m = Q_m{}^*$ for each $m=0, 1, \ldots, t$ provided that Δ, the determinant of the coefficients of $(Q_m - Q_m{}^*)$ in the system derived from (2.25), does not vanish. Clearly, Δ is a lower triangular determinant and is simply evaluated as the product of all the elements on its main diagonal. Thus,

$$\Delta = \prod_{m=0}^{t} (-)^m m!\,(\epsilon - m)_{t-m} \neq 0$$

under the conditions on ϵ given after (2.24).

Consider the representation formula

$$P_q(x) = \sum_{k=0}^{t} \frac{(-)^k(x+\gamma)_t(x+\gamma+t)}{k!\,(t-k)!\,(x+\gamma+k)} P_q(-\gamma-k), \tag{2.26}$$

which can be proved as follows: Since

$$\frac{(x+\gamma)_t(x+\gamma+t)}{x+\gamma+k} = (x+\gamma)_k(x+\gamma+k+1)_{t-k},$$

the right-hand side of (2.26) is a polynomial in x of degree t. By direct computation, the right-hand side of (2.26) agrees with $P_q(x)$ at the $t+1$ points $x=-\gamma-r$, $r=0, 1, \ldots, t$. Since $q \leqslant t$, this is sufficient to establish (2.26). To derive (2.23), we use Lemma A.1 proved in the Appendix with $n=t-k$, $\beta=(\epsilon-t)/2$ and $z=x+\gamma+t$. Thus

$$\begin{aligned} &{}_4F_3\left(\begin{matrix} k-t, x+\gamma+\epsilon, 1+\dfrac{\epsilon-t}{2}, 1 \\ \\ 1+\epsilon-k, 1-x-\gamma-t, \dfrac{\epsilon-t}{2} \end{matrix}\middle| 1\right) \\ &= \frac{(\epsilon-k)(x+\gamma+t)}{(\epsilon-t)(x+\gamma+k)}, \qquad (\epsilon-t)(x+\gamma+k)\neq 0. \end{aligned} \tag{2.27}$$

Now put $(x+\gamma+t)/(x+\gamma+k)$ from the latter formula into (2.26), express the ${}_4F_3$ as a sum over m from 0 to t, interchange summation processes and so obtain (2.23) with

$$Q_m = \frac{(t+\epsilon-2m)}{m!\,(\epsilon)_{t+1-m}} \sum_{k=0}^{m} \frac{(-m)_k(m-\epsilon-t)_k}{k!(1-\epsilon)_k} P_q(-\gamma-k). \tag{2.28}$$

Since

$$P_q(-\gamma-k)=(-)^q c\prod_{j=1}^{q}\frac{(\gamma+\omega_j)(1+\gamma+\omega_j)_k}{(\gamma+\omega_j)_k}, \tag{2.29}$$

(2.28) and (2.29) reduce to the first line of (2.24). Finally, to get the second line of (2.24), observe that

$$P_q(x)=\sum_{m=0}^{t}(x+\gamma^*)_m(x+\gamma^*+\epsilon^*)_{t-m}\,Q_m^*,$$

$$\gamma^*=\gamma+\epsilon,\quad \epsilon^*=-\epsilon,\quad Q_m^*=Q_{t-m}, \tag{2.30}$$

or

$$Q_m=\frac{(-1)^q(2m-\epsilon-t)(\gamma+\epsilon+\omega_q)\,c}{(t-m)!\,(-\epsilon)_{m+1}}$$
$$\times\,{}_{q+2}F_{q+1}\left(\begin{matrix}m-t,-m+\epsilon,1+\gamma+\epsilon+\omega_q\\ \\ 1+\epsilon,\gamma+\epsilon+\omega_q\end{matrix}\middle|\,1\right).$$

Turning this last series expression for Q_m around, we arrive at the second line of (2.24), which completes the proof of the lemma.

Under our tentative assumption that no α_k is equal to any β_h, the preceding lemma determines the $A_m(n,\lambda)$'s and $B_m(n,\lambda)$'s uniquely. This is sufficient to establish the last statement of the theorem. To see this explicitly, we note that $A_t(n,\lambda)\neq 0$ and assume that one of the functions of interest, $\Phi_n(z,\lambda)$, satisfies a difference equation of lower order than that specified, i.e.,

$$\sum_{m=0}^{t'}[A_m'(n,\lambda)+zB_m'(n,\lambda)]\Phi_{n-m}(z,\lambda)=R_n(z,\lambda),\qquad A_0'(n,\lambda)=1 \tag{2.31}$$

but with $t'<t$. If in this assumed equation, (2.31), we replace n by $n-1$, multiply the resulting equation by an arbitrary parameter ρ_1 and add it to the original equation, (2.31), we would then obtain an equation of the form (2.31), but with t' replaced by $t'+1$. After $t-t'$ repetitions of this process, the resulting equation would still be of the form (2.31), but with t' replaced by t. Since, by the lemma, the $A_m(n,\lambda)$'s are unique, we would have, in particular,

$$A_t(n,\lambda)=\rho_{t-t'}A_{t'}'(n-t,\lambda);\qquad \rho_{t-t'}\ \text{arbitrary},$$

which would contradict the nonzero uniqueness of $A_t(n,\lambda)$. Finally, our tentative assumption that no α_k is equal to any β_h can be relaxed completely by an appeal to continuity. The only penalty exacted for such a relaxation is that the $A_m(n,\lambda)$'s and $B_m(n,\lambda)$'s are no longer unique, and that the recurrence formulae are no longer of the lowest possible order. This completes the proof of the theorem.

COROLLARY 2.1. *The function*

$$\mathscr{E}(n,z,\lambda)=\frac{\Gamma(\beta_s)}{\Gamma(-n)\,\Gamma(n+\lambda)\,\Gamma(\alpha_r)}\,G^{1,r+3}_{r+3,s+1}\left(-z\,\middle|\,\begin{matrix}0,1+n,1-n-\lambda,1-\alpha_r\\ 0,1-\beta_s\end{matrix}\right) \tag{2.32}$$

satisfies the difference equation (2.10), *and if no α_k is equal to any β_h, satisfies no nontrivial equation of the same form, of lower order than t.*

Proof. Under conditions $C_{0,\lambda}$, $\mathscr{E}(n,z,\lambda)=\mathscr{E}_n(z,\lambda)$, while under conditions $C_{\infty,\lambda}$, it follows from (1.4) that $\mathscr{E}(n,z,\lambda)$ differs from $\mathscr{K}_n(z,\lambda)$ by a linear combination of the functions $\mathscr{G}_{n,k}(z,\lambda)$ whose coefficients are independent of both z and n, except possibly for a periodic function of n which has period unity.

Remark 2.1. The determination of the $A_m(n,\lambda)$'s and $B_m(n,\lambda)$'s was first given by Wimp [5] in the special case that one of the β_h's is unity. He gives two proofs. One of these is algebraic and essentially stems from the solution of the linear equation systems derived from (2.18) and (2.19) when one puts therein $v=1-\beta_h$, $h=1,\ \ldots,\ s$ and $w=1+n,\ 1-n-\lambda+t,\ 1-\alpha_k,\ k=1,\ \ldots,\ r$, respectively. The other proof shows that if the $A_m(n,\lambda)$'s and $B_m(n,\lambda)$'s are as given, they can be represented by contour integrals and that $\mathscr{E}_n(z,\lambda)$ with one of the β_h's equal to unity satisfies the then homogeneous difference equation (2.10).

Remark 2.2. The generalized Jacobi function $\mathscr{E}_n(z,\lambda)$ will lose its specialized appearance, if we let $s=q+1$ and set $\beta_h=1$, $h=q+1$.

Remark 2.3. As previously noted, under conditions $C_{0,\lambda}$ ($C_{\infty,\lambda}$), the functions $\mathscr{E}_{n,h}(z,\lambda)$ ($\mathscr{G}_{n,k}(z,\lambda)$), form a basis of the differential equation (2.5), and hence are linearly independent as functions of z. Although we have shown that these same functions satisfy the difference equation (2.13), it is not known whether they form a basis of solutions of (2.13), i.e., whether they are linearly independent as functions of n.

Remark 2.4. If the parameter condition in $C_{0,\lambda}$ or $C_{\infty,\lambda}$ is violated, additional solutions of the difference equation (2.13) can be found via the same limit processes used to find additional solutions of the differential equation (2.5).

Remark 2.5. If n is a nonnegative integer, no restrictions on r, s and z are necessary for $\mathscr{E}_n(z,\lambda)$ to be well defined and for the results of Theorem 2.1 to hold. In this connection, care must be taken in certain limit processes which may arise. For example, suppose $r=0$, $s=1$, $\beta_1=1$. Then from (2.11), we have $A_1(n,\lambda)=-(2n+\lambda-2)(\lambda-1)[(n+\lambda-1)(2n+\lambda-3)]^{-1}$. If we want

$A_1(n, \lambda)$ for $n = 1$ and $\lambda = 1$, we must first set $n = 1$ and then let $\lambda \to 1$. Thu $A_1(n, \lambda) = -1$ for $n = 1$ and $A_1(n, \lambda) = 0$ for $n \neq 1$.

Consider the limit procedure (called a confluence with respect to λ)

$$\lim_{\lambda\to\infty} \mathscr{E}_n(z/\lambda, \lambda) = \lim_{\lambda\to\infty} {}_{r+3}F_s\left(\begin{matrix} -n, n+\lambda, \alpha_r, 1 \\ \\ \beta_s \end{matrix}\middle| \frac{z}{\lambda}\right),$$

$$= {}_{r+2}F_s\left(\begin{matrix} -n, \alpha_r, 1 \\ \\ \beta_s \end{matrix}\middle| z\right) = \mathscr{E}_n(z), \tag{2.33}$$

which is valid for $r + 2 \leqslant s$. Thus, results for the generalized Laguerre functior can be deduced from those for the generalized Jacobi functions. In fact, if w write down the conditions

$$\left.\begin{array}{l} r+2 \leqslant s \quad \text{or} \quad r+1 = s \quad \text{and} \quad |\arg(1-z)| < \pi, \\ \text{no two of the parameters, } \beta_h\ (h = 1, \ldots, s), \text{ differ by an} \\ \text{integer,} \end{array}\right\} C$$

$$\left.\begin{array}{l} r \geqslant s \quad \text{or} \quad r+1 = s \quad \text{and} \quad |\arg(1-1/z)| < \pi, \\ \text{no two of the parameters, } -n,\ \alpha_k\ (k = 1, \ldots, r), \text{ differ by an} \\ \text{integer,} \end{array}\right\} C$$

and let

$$\lim_{\lambda\to\infty} \Phi_n(z/\lambda, \lambda) = \Phi_n(z),$$

$$\Phi_n(z, \lambda) = \mathscr{K}_n(z, \lambda), \quad \mathscr{F}_{n,h}(z, \lambda)\ (h = 1, \ldots, s), \quad \mathscr{G}_{n,k}(z, \lambda)\ (k = 1, \ldots, r+1), \tag{2.34}$$

a limiting form of Theorem 2.1 and Corollary 2.1 is the following.

COROLLARY 2.2. *The functions $\mathscr{E}_n(z)$ and $\mathscr{K}_n(z)$ under the conditions on r, and z in C_0 and C_∞, respectively, and the function*

$$\mathscr{E}(n, z) = \frac{\Gamma(\beta_s)}{\Gamma(-n)\,\Gamma(\alpha_r)}\, G^{1,\,r+2}_{r+2,\,s+1}\left(-z\,\middle|\begin{matrix} 0, 1+n, 1-\alpha_r \\ \\ 0, 1-\beta_s \end{matrix}\right) \tag{2.3}$$

satisfy the difference equation

$$\Phi_n(z) + \sum_{m=1}^{\bar{t}} [A_m(n) + zB_m(n)]\,\Phi_{n-m}(z) = \frac{(\beta_s - 1)}{(n + \beta_s - 1)},$$

$$\bar{t} = \max(r+1, s), \tag{2.3}$$

where

$$A_m(n) = \frac{(n+1-m)_m(n-m-1+\beta_s)}{m!\,(n-1+\beta_s)}\, {}_{s+1}F_s\left(\begin{matrix}-m, n-m+\beta_s \\ n-m-1+\beta_s\end{matrix}\middle| 1\right),$$

$$= \frac{(n+1-m)_m(-)^m}{m!\,(n-1+\beta_s)} \sum_{u=0}^{s-m} \frac{(s-u)!}{(s-m-u)!} B^{(-m)}_{s-m-u} S_u(n-m-1+\beta_s), \qquad (2.37)$$

$$B_m(n) = \frac{(n+1-m)_m(n-m+\alpha_r)}{(m-1)!\,(n-1+\beta_s)}\, {}_{r+1}F_r\left(\begin{matrix}1-m, n-m+1+\alpha_r \\ n-m+\alpha_r\end{matrix}\middle| 1\right),$$

$$= \frac{(n+1-m)_m(-)^{m-1}}{(m-1)!\,(n-1+\beta_s)} \sum_{u=0}^{r+1-m} \frac{(r-u)!}{(r+1-m-u)!} B^{(1-m)}_{r+1-m-u} S_u(n-m+\alpha_r), \qquad (2.38)$$

where $B_k^{(a)}$ *is the generalized Bernoulli number defined in* (A.7), *and where the* $S_u(\rho_q)$ *are the symmetric polynomials defined implicitly by*

$$\prod_{i=1}^{q} (x+\rho_i) = \sum_{u=0}^{q} S_u(\rho_q)\, x^{q-u}. \qquad (2.39)$$

In particular, $A_m(n) = 0$, $m \geqslant s+1$ *and* $B_m(n) = 0$, $m \geqslant r+2$. *In addition, the functions* $\mathscr{F}_{n,h}(z)$ $(h = 1, \ldots, s)$ and $\mathscr{G}_{n,k}(z)$ $(k = 1, \ldots, r+1)$ *under the conditions* C_0 *and* C_∞, *respectively, satisfy the difference equation*

$$\Phi_n(z) + \sum_{m=1}^{\bar{t}} [A_m(n) + zB_m(n)]\Phi_{n-m}(z) = 0. \qquad (2.40)$$

Finally, if no α_k *is equal to any* β_h, *none of the above functions satisfies a non-trivial equation of the form specified of lower order than* $\bar{t}$.

Remark 2.6. The only part of Corollary 2.2 which does not follow directly from Theorem 2.1 is the last statement, which is actually concerned with the uniqueness of

$$A_m(n) = \lim_{\lambda\to\infty} A_m(n, \lambda), \qquad B_m(n) = \lim_{\lambda\to\infty} \frac{B_m(n, \lambda)}{\lambda}.$$

The uniqueness of the $A_m(n)$ and $B_m(n)$ follows from the limiting form of Lemma 2.1 when $\epsilon \to \infty$, i.e.,

LEMMA 2.2. *If* $P_q(x)$ *is a polynomial in* x *of degree* q,

$$P_q(x) = c \prod_{i=1}^{q} (x - \omega_i), \qquad (2.41)$$

and t is an integer $\geqslant q$, then $P_q(x)$ can be represented uniquely in the form

$$P_q(x) = c \sum_{m=0}^{t} (x+\gamma)_m \bar{Q}_m, \tag{2.42}$$

$$\begin{aligned}\bar{Q}_m &= \frac{(-)^q(\gamma+\omega_q)\,c}{m!}\;{}_{q+1}F_q\left(\begin{matrix}-m, 1+\gamma+\omega_q\\ \\ \gamma+\omega_q\end{matrix}\middle|\,1\right),\\ &= \frac{(-)^{q-m}\,c}{m!}\sum_{u=0}^{q-m}\frac{(q-u)!}{(q-m-u)!}B_{q-m-u}^{(-m)}S_u(\gamma+\omega_q),\end{aligned} \tag{2.43}$$

where $B_k^{(a)}$ is the generalized Bernoulli number defined in (A.7) *and where the $S_u(\rho_q)$ are the symmetric polynomials defined implicitly by* (2.39). *Note that $\bar{Q}_m = 0$ for $m \geqslant q+1$. If $(\gamma+\omega_q)$ is zero, limits must be taken in* (2.43).

Remark 2.7. The second lines of (2.37, 38, 43) follow by an application of Lemma A.2. in the Appendix.

Remarks similar to those following Corollary 2.1 can also be made for Corollary 2.2.

To illustrate the principal results of this section, we have for n a positive integer that

$$\mathscr{P}_n(z,\lambda) = {}_4F_3\left(\begin{matrix}-n, n+\lambda, \alpha_1, \alpha_2\\ \\ \beta_1, \beta_2, \beta_3\end{matrix}\middle|\,z\right) \tag{2.44}$$

satisfies the difference equation

$$\Phi_n(z,\lambda) + \sum_{m=1}^{4}[A_m(n,\lambda) + zB_m(n,\lambda)]\Phi_{n-m}(z,\lambda) = 0, \qquad B_4(n,\lambda) = 0, \tag{2.45}$$

where

$$A_1(n,\lambda)$$
$$= \frac{(n-1)(2n+\lambda-2)_2(n-2+\beta)}{(n+\lambda-1)(2n+\lambda-5)(n-1+\beta)}\left\{1 - \frac{(2n+\lambda-5)(n)(n-1+\beta)}{(2n+\lambda-1)(n-1)(n-2+\beta)}\right\},$$

$$A_2(n,\lambda)$$
$$\begin{aligned}&= \frac{(n-2)_2(2n+\lambda-4)_4(n-3+\beta)}{2(n+\lambda-2)_2(2n+\lambda-6)_2(n-1+\beta)}\left\{1 - \frac{2(2n+\lambda-6)(n-1)(n-2+\beta)}{(2n+\lambda-3)(n-2)(n-3+\beta)}\right.\\ &\qquad \left. + \frac{(2n+\lambda-6)_2(n)(n-1+\beta)}{(2n+\lambda-3)_2(n-2)(n-3+\beta)}\right\},\end{aligned}$$

$$(n, \lambda)$$

$$\frac{n-2)_2(n+\lambda-4)(2n+\lambda-6)(2n+\lambda-3)_3(n+\lambda-3-\beta)}{(n+\lambda-3)_3(2n+\lambda-7)_3(n-1+\beta)}$$

$$\times\left\{1-\frac{(2n+\lambda-7)(n+\lambda-3)(n+\lambda-2-\beta)}{(2n+\lambda-3)(n+\lambda-4)(n+\lambda-3-\beta)}\right\},$$

$$(2.46)$$

$$(n, \lambda)$$

$$\frac{n-3)_3(n+\lambda-4)(2n+\lambda-8)(2n+\lambda-3)_3(n+\lambda-3-\beta)}{(n+\lambda-4)_4(2n+\lambda-8)_4(n-1+\beta)},$$

$$1+\sum_{m=1}^{4} A_m(n,\lambda)=0,$$

$$(n, \lambda)$$

$$\frac{2n+\lambda-2)_2(n-1+\alpha)}{(n+\lambda-1)(n-1+\beta)},$$

$$(n, \lambda)$$

$$\frac{(n-1)(2n+\lambda-4)_4(n-2+\alpha)}{n+\lambda-2)_2(2n+\lambda-5)(n-1+\beta)}\left\{1-\frac{(2n+\lambda-5)(n-1+\alpha)}{(2n+\lambda-3)(n-2+\alpha)}\right\},$$

$$(n, \lambda)$$

$$\frac{n-2)_2(2n+\lambda-3)_3(n+\lambda-3-\alpha)}{n+\lambda-3)_3(2n+\lambda-5)(n-1+\beta)}.$$

re $(n+u+\beta)$ is short for

$$\prod_{j=1}^{3}(n+u+\beta_j)$$

d $(n+u+\alpha)$ is short for

$$\prod_{j=1}^{2}(n+u+\alpha_j).$$

Similar recurrence formulae for any hypergeometric function of lower order n $\mathscr{P}_n(z,\lambda)$ can be found by taking limiting forms of (2.45) and (2.46). In rticular, recurrence relationships for

$$ {}_4F_2\left(\begin{matrix}-n, n+\lambda, \alpha_1, \alpha_2\\ \beta_1, \beta_2\end{matrix}\middle| z\right); \qquad {}_3F_3\left(\begin{matrix}-n, \alpha_1, \alpha_2\\ \beta_1, \beta_2, \beta_3\end{matrix}\middle| z\right),$$

may be found by replacing z in (2.45) by $z\beta_3$; z/λ and letting β_3; $\lambda \to \infty$. If bo
numerator and denominator parameters are to be removed from $\mathscr{P}_n(z,\lambda)$, ca
must be taken to obtain the nontrivial recurrence relation of lowest ord
e.g., to obtain the recurrence relation for

$$ {}_3F_2\left(\begin{matrix} -n, n+\lambda, \alpha_1 \\ \\ \beta_1, \beta_2 \end{matrix}\middle| z\right) $$

from (2.45), we set $\alpha_2 = \beta_3 = n + \lambda + 1 - t$, $t = 4$, so that $A_4(n,\lambda) = B_3(n,\lambda) =$
Recurrence formulae for special cases of the above have been given by Fase
meyer [*12*], and Rainville [*13*].

III. Recursion Formulae for the Numerator Polynomials in the Rational Approximations for the Generalized Hypergeometric Function

Suppose

$$ F(z) = \sum_{r=0}^{n-1} f_r z^r + R_n(z), \qquad (3 $$

where f_r is independent of n and z. In (3.1), replace n by $k+1-a$, $a = 0$ or
multiply both sides by $A_{n,k}\gamma^k$, $A_{n,k} = 0$ if $k > n$, and sum from $k = 0$ to $k =$
Then, just as in (1.9), (1.10), we obtain

$$ F(z)\, h_n(\gamma) = \psi_n(z,\gamma) + S_n(z,\gamma), \qquad (3 $$

$$ h_n(\gamma) = \sum_{k=0}^{n} A_{n,k}\gamma^k, \qquad (3 $$

$$ \psi_n(z,\gamma) = \sum_{k=a}^{n} \gamma^k \sum_{r=0}^{n-k} A_{n,r+k}\, f_r (z\gamma)^r, \qquad (3 $$

which can be interpreted as giving a formal rational approximation to $F($
i.e., $\psi_n(z,\gamma)/h_n(\gamma)$. As in (1.10), alternate expressions can be given for $\psi_n(z,$
In particular, $\psi_n(z,\gamma)$ is a weighted sum of partial sums of $F(z)$. We now pro
that if $h_n(\gamma)$ satisfies a linear, inhomogeneous recursion relation, then $\psi_n(z,$
satisfies another linear, inhomogeneous recursion relation and the hom
geneous portions of these recurrence relations are identical. This general res
is embodied in the following theorem.

Theorem 3.1. *Let $h_n(\gamma)$, $\psi_n(z,\gamma)$ be defined as above. Let there exist consta
$K_m(n)$, $L_m(n)$ and $M(n)$ such that*

$$ \sum_{m=0}^{t} [K_m(n) + \gamma L_m(n)]\, h_{n-m}(\gamma) = M(n), $$

$$ K_0(n) = 1, \qquad L_0(n) = 0, \qquad n \geqslant t. \qquad (3 $$

Then

$$Q_n(z,\gamma) = \sum_{m=0}^{t} [K_m(n) + \gamma L_m(n)]\, \psi_{n-m}(z,\gamma)$$

$$= \gamma^a \sum_{r=0}^{n-a} f_r(z\gamma)^r \sum_{m=0}^{t} K_m(n)\, A_{n-m,\, r+a}. \tag{3.6}$$

Further, if

$$h_n(\gamma) = {}_{r+3}F_s\left(\begin{matrix} -n, n+\lambda, \alpha_r, 1 \\ \beta_s \end{matrix}\middle|\, \gamma\right), \tag{3.7}$$

and (3.5) *is identified with* (2.10), *then*

$$Q_n(z,\gamma) = [-\gamma n \alpha_r]^a [\beta_s - 1]^{1-a} \frac{(2n+\lambda-t)_t\, \Gamma(n+\lambda+a-t)}{(n+\beta_s-1)\, \Gamma(n+\lambda)}$$

$$\times \sum_{j=0}^{n-a} \frac{(a-n)_j (n+\lambda+a-t)_j (\alpha_r+a)_j\, f_j(z\gamma)_j}{(\beta_s+a-1)_j},$$

$$t = \max(r+2, s). \tag{3.8}$$

Proof. Combining the first expression of (3.6) with (3.4), we have

$$Q_n(z,\gamma) = \sum_{m=0}^{t} [K_m(n) + \gamma L_m(n)] \sum_{k=a}^{n-m} \gamma^k \sum_{r=0}^{n-m-k} A_{n-m,\, r+k}\, f_r(z\gamma)^r.$$

Observe that we can replace the upper limits of the r and k summation indices by $n-a$ and n, respectively, since $A_{n,k} = 0$ if $k > n$. Thus, we can write

$$Q_n(z,\gamma) = \sum_{m=0}^{t} [K_m(n) + \gamma L_m(n)] \sum_{k=a}^{n} \gamma^k \sum_{r=0}^{n-a} A_{n-m,\, r+k}\, f_r(z\gamma)^r$$

$$= \gamma^a \sum_{r=0}^{n-a} f_r(z\gamma)^r \sum_{m=0}^{t} K_m(n)\, A_{n-m,\, r+a}$$

$$+ \sum_{k=a+1}^{n} \gamma^k \sum_{r=0}^{n-a} f_r(z\gamma)^r \sum_{m=0}^{t} [K_m(n)\, A_{n-m,\, r+k} + L_m(n)\, A_{n-m,\, r+k-1}].$$

But in view of (3.3) and (3.5)

$$\sum_{m=0}^{t} [K_m(n)\, A_{n-m,\, j} + L_m(n)\, A_{n-m,\, j-1}] = 0,$$

for j a positive integer. It follows that $Q_n(z,\gamma)$ is given by the second line of (3.6).

Next, we turn to the proof of (3.8). With $A_{n,k}$ defined by (3.7), and

$$C_{n,k} = \frac{(-n)_k(n+\lambda-t)_k(\alpha_r)_k}{(\beta_s)_k},$$
$$= \frac{(-n)_m(n+\lambda-t)_{t-m}}{(k-n)_m(k+n+\lambda-t)_{t-m}} A_{n-m,k},$$

it follows from (2.15, 18) that

$$\sum_{m=0}^{t} K_m(n) A_{n-m,k} = C_{n,k} X_t(k),$$
$$= \frac{(\beta_s-1)(n+\lambda)_n(-n)_k(n+\lambda-t)_k(\alpha_r)_k}{(n+\beta_s-1)(n+\lambda-t)_n(\beta_s-1)_k}. \tag{3.9}$$

Combining (3.9) with (3.6), we arrive at (3.8). This concludes the proof of the theorem.

The following corollary summarizes these results in a form convenient for applications.

COROLLARY 3.1. *If* $a = 0$ *or* 1,

$$h_n(\gamma) = {}_{f+q+3}F_{g+p+1}\left(\begin{matrix}-n, n+\lambda, \sigma_f, \beta_q - a, 1\\ \beta+1, \rho_g, \alpha_p+1-a\end{matrix}\middle|\gamma\right), \tag{3.10}$$

$$\psi_n(z,\gamma) = \left[-\frac{\gamma n(n+\lambda)\sigma_f(\beta_q-1)}{(\beta+1)\rho_g\alpha_p}\right]^a$$
$$\times \sum_{k=0}^{n-a} \frac{(-n+a)_k(n+\lambda+a)_k(\sigma_f+a)_k(\alpha_p)_k(\gamma z)^k}{(\beta+1+a)_k(\rho_g+a)_k(1+\alpha_p)_k\, k!}$$
$$\times {}_{f+q+3}F_{g+p+1}\left(\begin{matrix}-n+k+a, n+\lambda+k+a, \sigma_f+k+a, \beta_q+k, 1\\ \beta+1+k+a, \rho_g+k+a, 1+\alpha_p+k\end{matrix}\middle|\gamma\right) \tag{3.11}$$

and $t = \max\{f+q+2, g+p+1\}$,

$$K_m(n,\lambda)$$
$$= \frac{(n+1-m)_m(2n+\lambda-2m)_{2m}(n-m+\beta)(n-m+\rho_g-1)(n-m+\alpha_p-a)}{m!\,(n+\lambda-m)_m(2n+\lambda-t-m)_m(n+\beta)(n+\rho_g-1)(n+\alpha_p-a)}$$
$$\times {}_{g+p+3}F_{g+p+2}\left(\begin{matrix}-m, 2n+\lambda-t-m, n-m+\beta+1, n-m+\rho_g, n-m+\alpha_p+1-a\\ 2n+\lambda+1-2m, n-m+\beta, n-m+\rho_g-1, n-m+\alpha_p-a\end{matrix}\middle|1\right),$$

$$= (-1)^{g+p+1} \frac{(n+1-m)_m(2n+\lambda-2m)(2n+\lambda-t+1)_{t-1}(n+\lambda-t-\beta)(n+\lambda-t+1-\rho_g)(n+\lambda-t-\alpha_p+a)}{(t-m)!\,(n+\lambda-m)_m(2n+\lambda-t-m)_m(n+\beta)(n+\rho_g-1)(n+\alpha_p-a)}$$

$$\times {}_{g+p+3}F_{g+p+2}\left(\begin{matrix}-t+m, 2n+\lambda-t-m, n+\lambda-t+1-\beta, n+\lambda-t+2-\rho_g, n+\lambda-t+1-\alpha_p+a \\ 2n+\lambda+1-t, n+\lambda-t-\beta, n+\lambda-t+1-\rho_g, n+\lambda-t-\alpha_p+a\end{matrix}\middle| 1\right), \tag{3.12}$$

$_m(n, \lambda)$

$$= \frac{(n+1-m)_m(2n+\lambda-2m)_{2m}(n-m+\sigma_f)(n-m+\beta_q-a)}{(m-1)!\,(n+\lambda-m)_m(2n+\lambda-t-m+1)_{m-1}(n+\beta)(n+\rho_g-1)(n+\alpha_p-a)}$$

$$\times {}_{f+q+2}F_{f+q+1}\left(\begin{matrix}1-m, 2n+\lambda-t-m+1, n-m+1+\sigma_f, n-m+1+\beta_q-a \\ 2n+\lambda+1-2m, n-m+\sigma_f, n-m+\beta_q-a\end{matrix}\middle| 1\right),$$

$$= (-1)^{f+q} \frac{(n+1-m)_m(2n+\lambda-2m)(2n+\lambda-t+1)_{t-1}(n+\lambda-t+1-\sigma_f)(n+\lambda-t+1-\beta_q+a)}{(t-m-1)!\,(n+\lambda-m)_m(2n+\lambda-t-m+1)_{m-1}(n+\beta)(n+\rho_g-1)(n+\alpha_p-a)}$$

$$\times {}_{f+q+2}F_{f+q+1}\left(\begin{matrix}-t+m+1, 2n+\lambda-t-m+1, n+\lambda-t+2-\sigma_f, n+\lambda-t+2-\beta_q+a \\ 2n+\lambda+1-t, n+\lambda-t+1-\sigma_f, n+\lambda-t+1-\beta_q+a\end{matrix}\middle| 1\right), \tag{3.13}$$

hen $\psi_n(z,\gamma)/h_n(\gamma)$ *is a formal rational approximation to*

$$ {}_pF_q\left(\begin{matrix}\alpha_p \\ \beta_q\end{matrix}\middle| z\right),$$

uch that

$$\sum_{m=0}^{t} [K_m(n,\lambda) + \gamma L_m(n,\lambda)]\, h_{n-m}(\gamma) = \frac{\beta(\rho_g-1)(\alpha_p-a)(n+\lambda)_n}{(n+\beta)(n+\rho_g-1)(n+\alpha_p-a)(n+\lambda-t)_n}, \tag{3.14}$$

nd

$$\sum_{m=0}^{t} [K_m(n,\lambda) + \gamma L_m(n,\lambda)]\, \psi_{n-m}(z,\gamma)$$

$$= [-\gamma n \sigma_f(\beta_q-1)]^a [\beta(\rho_g-1)\,\alpha_p]^{1-a} \frac{(2n+\lambda-t)_t\, \Gamma(n+\lambda+a-t)}{(n+\beta)(n+\alpha_p-a)(n+\rho_g-1)\,\Gamma(n+\lambda)}$$

$$\times {}_{f+2}F_{g+1}\left(\begin{matrix}-n+a, n+\lambda-t+a, \sigma_f+a \\ \beta+a, \rho_g-1+a\end{matrix}\middle| \gamma z\right). \tag{3.15}$$

he difference equations (3.14) *and* (3.15) *are homogeneous if*

$$\beta(\rho_g-1)(\alpha_p-a) = 0 \qquad \text{and} \qquad [\sigma_f(\beta_q-1)]^a = 0,$$

espectively. In the special case $\gamma z = 1$, $f = g = 0$, $\lambda = \alpha + \beta + 1$, *Eq.* (3.15) *is lso homogeneous, if* $2 + a + \alpha - t$ *is a nonpositive integer.*

Remark 3.1. Note that the ${}_{f+2}F_{g+1}$ on the right-hand side of (3.15) is a extended Jacobi polynomial and so may be generated by application of th recursion formula developed in Section II.

Remark 3.2. Care must be taken in (3.15) if $a = 0$ and $\beta(\rho_g - 1)$ approache zero. In particular, one must use the relation

$$\beta(\rho_g - 1)\, {}_{f+2}F_{g+1}\left(\begin{matrix}-n, n+\lambda-t, \sigma_f \\ \beta, \rho_g - 1\end{matrix}\middle| \gamma z\right)$$
$$= \beta(\rho_g - 1)$$
$$+ (-n)(n+\lambda-t)(\sigma_f)(\gamma z)\, {}_{f+3}F_{g+2}\left(\begin{matrix}1-n, 1+n+\lambda-t, 1+\sigma_f, 1 \\ 1+\beta, \rho_g, 2\end{matrix}\middle| \gamma z\right) \quad (3.16$$

Remark 3.3. Should a numerator parameter σ be equal to a denominato parameter ρ in (3.10), (3.14) and (3.15) will in general reduce in length only i $\sigma = \rho = n + \lambda + 1 - t$. For particular numerical values of $\sigma = \rho\, (\neq n + \lambda + 1 - t$ (3.14) and (3.15), though still valid, are no longer the desired recursion formula of shortest length.

Remark 3.4. Confluent forms of Theorem 3.1 and Corollary 3.1 are easil found by replacing γ by γ/λ, and letting $\lambda \to +\infty$.

Remark 3.5. An alternate technique for the evaluation of $\psi_n(z, \gamma)$ as definec by (3.11) is the following: if

$$V_{n,k}(\gamma) = {}_{r+3}F_s\left(\begin{matrix}-n+a+k, n+\lambda+a+k, \theta_r+k, 1 \\ \omega_s + k\end{matrix}\middle| \gamma\right), \quad (3.17$$

then

$$V_{n,k-1}(\gamma) = 1 + \gamma \frac{(-n+a-1+k)(n+\lambda+a-1+k)(\theta_r - 1 + k)}{(\omega_s - 1 + k)} V_{n,k}(\gamma),$$
$$V_{n,n-a}(\gamma) = 1. \quad (3.18$$

IV. Recursion Formulae for Particular G-Functions

In this section, we obtain a linear, homogeneous difference equation for th G-functions

$$\mathscr{V}(n, \omega, \lambda) = G^{m,l}_{p+1,q+2}\left(\omega \middle| \begin{matrix} a_{p+1} \\ b_q, n, -n-\lambda\end{matrix}\right),$$
$$0 \leqslant m \leqslant q, \qquad 0 \leqslant l \leqslant p+1. \quad (4.1$$

early, $\mathscr{V}(n,\omega,\lambda)$ includes those G-functions occurring in (1.16) as coefficients the generalized Jacobi polynomials. As in Section II, properly normalized ses of

$$-1)^{p-m-l}\,\omega(\delta+1-a_{p+1})+(\delta-b_q)(\delta-n)(\delta+n+\lambda)\}\,Y(\omega)=0,$$

$$\delta=\omega\frac{d}{d\omega}, \tag{4.2}$$

equation satisfied by $\mathscr{V}(n,\omega,\lambda)$], also satisfy the above-mentioned difference uation for $\mathscr{V}(n,\omega,\lambda)$. For example, if $p>q+1$, the functions,

$$\mathscr{U}_h(n,\omega,\lambda)=G^{1,\,p+1}_{p+1,\,q+2}\left(\omega e^{-i\pi(m+l-p)}\,\middle|\,\begin{matrix}a_{p+1}\\ b_h,b_1,\ldots,b_{h-1},b_{h+1},\ldots,b_q,n,-n-\lambda\end{matrix}\right),$$

$$h=1,\ldots,q,$$

$$_{+1}(n,\omega,\lambda)=e^{i\phi n}\,G^{1,\,p+1}_{p+1,\,q+2}\left(\omega e^{-i\pi(m+l-p)}\,\middle|\,\begin{matrix}a_{p+1}\\ n,b_q,-n-\lambda\end{matrix}\right), \tag{4.3}$$

$$_{+2}(n,\omega,\lambda)=e^{i\phi n}\,G^{1,\,p+1}_{p+1,\,q+2}\left(\omega e^{-i\pi(m+l-p)}\,\middle|\,\begin{matrix}a_{p+1}\\ -n-\lambda,b_q,n\end{matrix}\right),$$

h $e^{i\phi}=-1$, and

$$\mathscr{S}_k(n,\omega,\lambda)=G^{0,\,p+1}_{p+1,\,q+2}\left(\omega e^{i\pi(2k+m+l-p-1)}\,\middle|\,\begin{matrix}a_{p+1}\\ b_q,n,-n-\lambda\end{matrix}\right),$$

$$k=1,\ldots,p-(q+1), \tag{4.4}$$

m the desired basis, normalized with respect to n, in a proper sector of the egular singular point $\omega=0$. Alternately, if $q+1>p$, the functions

$$\mathscr{H}_k(n,\omega,\lambda)=G^{q+2,\,1}_{p+1,\,q+2}\left(\omega e^{i\pi(q-m-l+1)}\,\middle|\,\begin{matrix}a_k,a_1,\ldots,a_{k-1},a_{k+1},\ldots,a_{p+1}\\ b_q,n,-n-\lambda\end{matrix}\right),$$

$$k=1,\ldots,p+1, \tag{4.5}$$

$$\mathscr{W}_h(n,\omega,\lambda)=G^{q+2,\,0}_{p+1,\,q+2}\left(\omega e^{-i\pi(2h+m+l-q)}\,\middle|\,\begin{matrix}a_{p+1}\\ b_q,-n,n+\lambda\end{matrix}\right),$$

$$h=1,\ldots,q+1-p, \tag{4.6}$$

form the desired basis, normalized with respect to n, in a proper sector of t
irregular singular point $\omega = \infty$. We tacitly assume that all of the above functio
are well defined.

We now state our main result of this section.

THEOREM 4.1. *Provided the functions* $\mathscr{V}(n,\omega,\lambda)$, $\mathscr{U}_h(n,\omega,\lambda)$, $\mathscr{S}_k(n,\omega,$
$\mathscr{H}_k(n,\omega,\lambda)$ *or* $\mathscr{W}_h(n,\omega,\lambda)$ *are well defined, they satisfy the difference equation*

$$\Psi(n,\omega,\lambda) + \sum_{j=1}^{t} [C_j(n,\lambda) + \omega^{-1} D_j(n,\lambda)]\, \Psi(n+j,\omega,\lambda) = 0,$$
$$t = \max(q+2, p+1), \qquad D_t(n,\lambda) = 0, \qquad (4.$$

where

$$C_j(n,\lambda) = \frac{(-1)^j (2n+\lambda+2j)(2n+\lambda)_j}{j!\,(2n+\lambda)}\, {}_{p+3}F_{p+2}\left(\begin{matrix} -j, j+2n+\lambda, n+2-a_{p+1} \\ 2n+\lambda+t+1, n+1-a_{p+1} \end{matrix}\middle|\, 1 \right.$$
$$= \frac{(-1)^{j+p+1}(2n+\lambda+1)_t (n+\lambda+j-1+a_{p+1})}{(t-j)!\,(2n+\lambda+j)_j (n+1-a_{p+1})}$$
$$\times\, {}_{p+3}F_{p+2}\left(\begin{matrix} j-t, 2n+\lambda+j, n+\lambda+j+a_{p+1} \\ 2n+\lambda+2j+1, n+\lambda+j-1+a_{p+1} \end{matrix}\middle|\, 1\right),$$
$$(4.$$

$$D_j(n,\lambda) = \frac{(-1)^{j+p+m+l+1}(2n+\lambda+2j)(2n+\lambda+1)_j (n+1-b_q)}{(j-1)!\,(n+1-a_{p+1})}$$
$$\times\, {}_{q+2}F_{q+1}\left(\begin{matrix} -j+1, 2n+\lambda+j+1, n+2-b_q \\ 2n+\lambda+t+1, n+1-b_q \end{matrix}\middle|\, 1\right),$$
$$= \frac{(-1)^{j+p+m+l+q+1}(2n+\lambda+j)(2n+\lambda+1)_t (n+\lambda+j+b_q)}{(t-j-1)!\,(2n+\lambda+j)_j (n+1-a_{p+1})}$$
$$\times\, {}_{q+2}F_{q+1}\left(\begin{matrix} j+1-t, 2n+\lambda+j+1, n+\lambda+j+1+b_q \\ 2n+\lambda+2j+1, n+\lambda+j+b_q \end{matrix}\middle|\, 1\right).$$
$$(4$$

*Moreover, if no a_k is equal to any b_h, none of the above functions satisfy a no
trivial equation of the form specified of lower order than t.*

Proof. Tentatively, we assume that no a_k is equal to any b_h and that ω
sufficiently restricted for the following manipulations to be valid. Consider t

nction $\mathscr{V}(n,\omega,\lambda)$. It follows from the integral representation for $\mathscr{V}(n,\omega,\lambda)$ e (1.3)], that, for $C_0(n,\lambda)=1$,

$$n,\omega,\lambda)$$

$$\sum_{=0}^{t} [C_j(n,\lambda)+\omega^{-1} D_j(n,\lambda)]\mathscr{V}(n+j,\omega,\lambda),$$

$$\frac{-1)^t}{2\pi i}\int_L \frac{\prod_{k=1}^{m}\Gamma(b_k-v)\prod_{k=1}^{l}\Gamma(1-a_k+v)\,\omega^v\sum_{j=0}^{t}(n-v)_j(-n-\lambda-t-v)_{t-j}\,C_j(n,\lambda)}{\prod_{k=m+1}^{q}\Gamma(1-b_k+v)\prod_{k=l+1}^{p+1}\Gamma(a_k-v)\,\Gamma(v+1-n)\,\Gamma(v+n+\lambda+t+1)}\,dv$$

$$+\frac{(-1)^t}{2\pi i}\int_{L-1} \frac{\prod_{k=1}^{m}\Gamma(b_k-1-v)\prod_{k=1}^{l}\Gamma(2-a_k+v)\,\omega^v\sum_{j=0}^{t}(n-1-v)_j(-n-\lambda-t-1-v)_{t-j}\,D_j(n,\lambda)}{\prod_{k=m+1}^{q}\Gamma(2-b_k+v)\prod_{k=l+1}^{p+1}\Gamma(a_k-1-v)\,\Gamma(v+2-n)\,\Gamma(v+n+\lambda+t+2)}\,dv, \tag{4.10}$$

iere the contour L, independent of j, runs parallel to the imaginary axis, and indented to separate the poles of $\Gamma(b_h-v)$ $(h=1,\ldots,m)$ from the poles of $1-a_k+v)$ $(k=1,\ldots,l)$. Let $R(v)$ denote the integrand of the first integral (4.10). Then, moving the contour of integration of the first integral in 10) one unit to the left, we obtain

$$n,\omega,\lambda)$$

$$\frac{-1)^{m+t}}{2\pi i}\int_{L-1} \frac{\prod_{k=1}^{m}\Gamma(b_k-1-v)\prod_{k=1}^{l}(1-a_k+v)\,\omega^v P(v)}{\prod_{k=m+1}^{q}\Gamma(2-b_k+v)\prod_{k=l+1}^{p+1}\Gamma(a_k-v)\,\Gamma(v+2-n)\,\Gamma(v+n+\lambda+t+2)}\,dv+R \tag{4.11}$$

iere

$$P(v)=(v+1-n)(v+n+\lambda+t+1)\prod_{h=1}^{q}(v+1-b_h)$$

$$\times\sum_{j=0}^{t}(n-v)_j(-n-\lambda-t-v)_{t-j}\,C_j(n,\lambda)$$

$$-(-1)^{p+l+m}\prod_{k=1}^{p+1}(v+1-a_k)$$

$$\times\sum_{j=0}^{t}(n-1-v)_j(-n-\lambda-t-1-v)_{t-j}\,D_j(n,\lambda),$$

d R is the sum of the residues due to those poles of $R(v)$ which lie between L d $L-1$, if any.

We now determine the $C_j(n,\lambda)$, $D_j(n,\lambda)$ by requiring that $P(v)=0$ for all v, that $\mathscr{Q}(n,\omega,\lambda)$ reduces to just R. Then, under the assumptions that

$t = \max(q+2, p+1)$ and $a_k \neq b_h$, it follows, just as in the proof of Theore
2.1, that there exists a constant C such that

$$\sum_{j=0}^{t} (n-v)_j(-n-\lambda-t-v)_{t-j}\, C_j(n,\lambda)$$

$$= C(-1)^{p+l+m} \prod_{k=1}^{p+1} (v+1-a_k),$$

(4.1

$$\sum_{j=0}^{t} (n-1-v)_j(-n-\lambda-t-1-v)_{t-j}\, D_j(n,\lambda)$$

$$= C(v+1-n)(v+n+\lambda+t+1) \prod_{h=1}^{q} (v+1-b_h).$$

Setting $v = n$ in the first line of (4.12), we find

$$C = (-1)^{t+p+l+m} \frac{(2n+\lambda+1)_t}{(n+1-a_{p+1})}. \quad (4.1$$

The values of $C_j(n,\lambda)$, $D_j(n,\lambda)$ given in (4.8), (4.9) then follow directly fr
(4.12, 13) and Lemma 2.1. It also follows from (4.12) that

$$R(v) = C(-1)^{m+1}$$

$$\times \frac{\prod_{k=1}^{m} \Gamma(b_k - v) \prod_{k=1}^{l} \Gamma(2-a_k+v)\, \omega^v}{\prod_{k=m+1}^{q} \Gamma(1-b_k+v) \prod_{k=l+1}^{p+1} \Gamma(a_k-1-v)\, \Gamma(v+1-n)\, \Gamma(v+n+\lambda+t+}$$

which clearly has no poles between L and $L-1$. Thus $\mathscr{Q}(n,\omega,\lambda) = R =$
Under our assumption that no a_k is equal to any b_h, the $C_j(n,\lambda)$, $D_j(n,\lambda)$
unique, which implies that $\mathscr{V}(n,\omega,\lambda)$ does not satisfy a nontrivial equation
the form specified of lower order than t. As before, the tentative assumptic
on a_k, b_h and ω can be relaxed completely by an appeal to continuity. Similar
the other functions of the theorem can be shown to satisfy (4.7). This comple
the proof.

Remark 4.1. The functions $e^{in\phi}\, \mathscr{V}(n,\omega,\lambda)$ with $m = q+1$, $e^{i\phi} = -1$, a
$\mathscr{V}(n,\omega,\lambda)$ with $m = q+2$ also satisfy (4.7).

Remark 4.2. Confluent limits can be taken in Theorem 4.1. In particul
recursion formulae for

$$\mathscr{V}(n,\omega) = \lim_{\lambda\to\infty} \Gamma(n+\lambda+1)\, \mathscr{V}(n,\omega\lambda,\lambda),$$

$$= G_{p+1,q+2}^{m,l}\left(\omega \middle| \begin{matrix} a_{p+1} \\ b_q, n \end{matrix}\right),$$

$$0 \leqslant l \leqslant p+1, \qquad 0 \leqslant m \leqslant q, \quad (4.$$

can be deduced from (4.7). The $\mathscr{V}(n,\omega)$ occur as coefficients in the generalized Laguerre expansion (1.17).

Remark 4.3. To expand an arbitrary hypergeometric function ${}_{p+1}F_q(\omega z)$, $p \leqslant q$, in a series of extended Jacobi polynomials, it follows from (1.18) that it is sufficient to consider

$$\mathscr{C}_n(\omega,\lambda) = \frac{(\alpha_{p+1})_n \omega^n}{(\beta_q)_n (n+\lambda)_n} {}_{p+1}F_{q+1}\left(\left.\begin{matrix} n+\alpha_{p+1} \\ n+\beta_q, 2n+\lambda+1 \end{matrix}\right| \omega\right),$$
$$= \frac{\Gamma(\beta_q)(2n+\lambda)\,\Gamma(n+\lambda)}{\Gamma(\alpha_{p+1})}(-1)^n G^{1,\,p+1}_{p+1,\,q+2}\left(-\omega\left|\begin{matrix} 1-\alpha_{p+1} \\ n, 1-\beta_q, -n-\lambda \end{matrix}\right.\right), \tag{4.15}$$

n a nonnegative integer, $p \leqslant q$, or $p = q+1$ and $|\arg(1-z)| < \pi$. Comparing (4.3) and (4.15), we see that

$$\mathscr{C}_n(\omega,\lambda) = \frac{\Gamma(\beta_q)(2n+\lambda)\,\Gamma(n+\lambda)}{\Gamma(\alpha_{p+1})} \mathscr{U}_{q+1}(n,\omega,\lambda) \tag{4.16}$$

with $l = p+1$, $m = 0$, $1-\alpha_{p+1} = a_{p+1}$ and $1-\beta_q = b_q$. Thus, recursion formulae for $\mathscr{C}_n(\omega,\lambda)$ can be deduced from Theorem 4.1. In particular,

$$\mathscr{Q}_n(\omega,\lambda) = \frac{(\alpha_1)_n(\alpha_2)_n(\alpha_3)_n(\alpha_4)_n}{(\beta_1)_n(\beta_2)_n(n+\lambda)_n}\omega^n\, {}_4F_3\left(\left.\begin{matrix} n+\alpha_1, n+\alpha_2, n+\alpha_3, n+\alpha_4 \\ n+\beta_1, n+\beta_2, 2n+\lambda+1 \end{matrix}\right|\omega\right) \tag{4.17}$$

satisfies the difference equation

$$\Phi_n(\omega,\lambda) + \sum_{j=1}^{4}[E_j(n,\lambda) + \omega^{-1}F_j(n,\lambda)]\Phi_{n+j}(\omega,\lambda) = 0, \qquad F_4(n,\lambda) = 0, \tag{4.18}$$

where

$$E_1(n,\lambda) = -\frac{(2n+\lambda)}{(n+\lambda)}\left\{1 - \frac{(2n+\lambda+1)(n+\alpha+1)}{(2n+\lambda+5)(n+\alpha)}\right\},$$

$$E_2(n,\lambda) = \frac{(2n+\lambda)_2}{2(n+\lambda)_2}\left\{1 - \frac{2(2n+\lambda+2)(n+\alpha+1)}{(2n+\lambda+5)(n+\alpha)} + \frac{(2n+\lambda+2)_2(n+\alpha+2)}{(2n+\lambda+5)_2(n+\alpha)}\right\},$$

$$E_3(n,\lambda) = -\frac{(2n+\lambda)_3(n+\lambda+3-\alpha)}{(n+\lambda)_3(2n+\lambda+5)_2(n+\alpha)}\left\{1 - \frac{(2n+\lambda+3)(n+\lambda+4-\alpha)}{(2n+\lambda+7)(n+\lambda+3-\alpha)}\right\},$$

$$E_4(n,\lambda) = \frac{(2n+\lambda)_4(n+\lambda+4-\alpha)}{(n+\lambda)_4(2n+\lambda+5)_4(n+\alpha)}, \tag{4.19}$$

$$F_1(n,\lambda) = -\frac{(2n+\lambda)_2(n+\beta)}{(n+\lambda)(n+\alpha)},$$

$$F_2(n,\lambda) = \frac{(2n+\lambda)_3(n+\beta)}{(n+\lambda)_2(n+\alpha)}\left\{1 - \frac{(2n+\lambda+3)(n+\beta+1)}{(2n+\lambda+5)(n+\beta)}\right\},$$

$$F_3(n,\lambda) = -\frac{(2n+\lambda)_4(n+\lambda+4-\beta)}{(n+\lambda)_3(2n+\lambda+5)_2(n+\alpha)}.$$

Here, $(n+u+\alpha)$ is short for

$$\prod_{j=1}^{4}(n+u+\alpha_j),$$

and $(n+u+\beta)$ is short for

$$\prod_{j=1}^{2}(n+u+\beta_j).$$

Similar recurrence formulae for any hypergeometric function of lower orde than $\mathcal{Q}_n(z,\lambda)$ can be found by taking limiting forms of (4.18) and (4.19).

Appendix

Lemma A.1.

$${}_4F_3\left(\begin{matrix}-n, \beta+1, 1, z+2\beta \\ n+2\beta+1, \beta, 1-z\end{matrix}\middle|\, 1\right) = \frac{z(n+2\beta)}{2\beta(z-n)},$$

$$n = 0, 1, 2, \ldots; \qquad \beta(z-n) \neq 0. \qquad \text{(A.1}$$

Proof. Let $V(z)$ equal $(-z)^{-1}$ times the hypergeometric function appearin on the left-hand side of (A.1). Clearly $V(z)$ is a rational function of z, with th degree of the denominator polynomial one greater than that of the numerato polynomial. Moreover, as the poles of $V(z)$ are simple and located a $0, 1, \ldots, n$, we can write

$$V(z) = \sum_{j=0}^{n} \frac{q_j}{z-i}$$

(A.2

$$q_j = \frac{(-1)^{j-1}(-n)_j(\beta+1)_j(2\beta+j)_j}{j!\,(n+2\beta+1)_j(\beta)_j}\,{}_3F_2\left(\begin{matrix}-n+j, \beta+1+j, 2j+2\beta \\ n+2\beta+1+j, \beta+j\end{matrix}\middle|\, 1\right).$$

hen, making use of the relation

$$F_2\left(\begin{matrix}-n+j, \beta+1+j, 2j+2\beta \\ n+2\beta+1+j, \beta+j\end{matrix}\middle| 1\right)$$

$${}_2F_1\left(\begin{matrix}-n+j, 2j+2\beta \\ n+2\beta+1+j\end{matrix}\middle| 1\right) - \frac{2(n-j)}{(n+2\beta+1+j)}\,{}_2F_1\left(\begin{matrix}-n+j+1, 2j+2\beta+1 \\ n+2\beta+2+j\end{matrix}\middle| 1\right), \tag{A.3}$$

gether with Gauss' theorem [*6*] for summing a ${}_2F_1$ of unit argument, we find

$$\begin{aligned} q_j &= (-1)\frac{(n+2\beta)}{2\beta}, \qquad j=n, \\ &= 0, \qquad j=0,1,\ldots,n-1, \end{aligned} \tag{A.4}$$

hich proves (A.1).

LEMMA A.2.

$$(-)^m(a_p)_1\ {}_{p+1}F_p\left(\begin{matrix}-m, 1+a_p \\ a_p\end{matrix}\middle| 1\right) = \sum_{j=0}^{p-m} \frac{(j+m)!}{j!} B_j^{(-m)} S_{p-m-j}(a_p), \tag{A.5}$$

here the $S_r(a_p)$ are the symmetric polynomials defined by

$$\prod_{r=1}^{p} (x+a_r) = \sum_{r=0}^{p} S_r(a_p)\, x^{p-r}, \tag{A.6}$$

d the $B_j^{(-m)}$ are the generalized Bernoulli numbers defined by

$$\left(\frac{e^t-1}{t}\right)^m = \sum_{j=0}^{\infty} \frac{t^j}{j!} B_j^{(-m)}, \qquad B_0^{(-m)} = 1. \tag{A.7}$$

$(a_p)_1$ is zero, limits must be taken in (A.5).

Proof. Let $\delta = xD = x(d/dx)$. It follows from the simple operator equation $)(\delta-1)\cdots(\delta-n+1) = x^2 D^n$, and the finite difference formula, see [*14*, 150, Eq. (90)],

$$x^k = \sum_{r=0}^{k} \binom{k}{r} B_{k-r}^{(-r)}\, x(x-1)\cdots(x-r+1), \qquad k=0,1,\ldots, \tag{A.8}$$

th x replaced by δ, that

$$\delta^k = \sum_{r=0}^{k} \binom{k}{r} B_{k-r}^{(-r)}\, x^r D^r, \tag{A.9}$$

or

$$\prod_{j=1}^{p}(\delta + a_j) = \sum_{m=0}^{p} x^m D^m \sum_{j=0}^{p-m} \binom{m+j}{m} B_j^{(-m)} S_{p-m-j}(a_p). \qquad \text{(A.1}$$

Now letting (A.10) operate on x^ρ, and making use of the operator equati $(\delta + \sigma)x^\rho = x^\rho(\rho + \sigma)$, we obtain

$$\prod_{j=1}^{p}(x + a_i) = \sum_{m=0}^{p} x(x-1)\cdots(x-m+1) \sum_{j=0}^{p-m} \binom{m+j}{m} B_j^{(-m)} S_{p-m-j}(a_p) \qquad \text{(A.1}$$

or

$$\sum_{j=0}^{p-m} \binom{m+j}{m} B_j^{(-m)} S_{p-m-j}(a_p) = \frac{1}{m!} \Delta^m \left\{ \prod_{j=1}^{p} (x + a_j) \right\} \Bigg|_{x=0}, \qquad \text{(A.1}$$

where Δ is the forward difference operator with respect to x. Since

$$\frac{1}{m!} \Delta^m \left\{ \prod_{j=1}^{p} (x + a_j) \right\} \Bigg|_{x=0} = \frac{(-)^m}{m!} \sum_{r=0}^{m} \frac{(-m)_r}{r!} \prod_{j=1}^{p} (r + a_j), \qquad \text{(A.1}$$

(A.12) reduces to (A.5).

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Recursion Formulae for Hypergeometric Functions

By Jet Wimp

I. Notation. The series definition for the generalized hypergeometric function is

$$(1)\qquad {}_PF_Q\left(\begin{matrix} a_P \\ b_Q \end{matrix}\middle|\, x\right) = \sum_{k=0}^{\infty} \frac{(a_P)_k x^k}{(b_Q)_k k!},$$

where

$$(2)\qquad (\alpha)_k = \Gamma(\alpha + k)/\Gamma(\alpha)$$

is Pochhammer's symbol and the shorthand product notation above will be used throughout this paper. In general, where a parameter has a subscript which is a capital letter, the repeated product notation is understood:

$$(3)\qquad (a_P)_k = \prod_{j=1}^{P} (a_j)_k, \qquad (n + b_Q) = \prod_{j=1}^{Q} (n + b_j), \qquad \text{etc.},$$

and the * notation

$$(4)\qquad (1 + b_Q - b_h)^* = \prod_{j=1;\ j\neq h}^{Q} (1 + b_j - b_h)$$

indicates the term corresponding to $j = h$ is to be deleted.

If one of the $a_i = 0$ or a negative integer, then (1) always converges, since it terminates. Otherwise it converges for all finite x if $P \leqq Q$ and for $|x| < 1$ if $P = Q + 1$. In this case, however, the function can be analytically continued into the cut plane $|\arg(1 - x)| < \pi$, and we shall often denote by ${}_{Q+1}F_Q(x)$ not only the series (1), whenever it converges, but also the analytic continuation of the series. If $P > Q + 1$, the series does not converge (unless it terminates) and if one of the b_j is 0 or a negative integer, the series is not defined. If one of the a_i equals one of the b_j, ${}_PF_Q(x)$ reduces to ${}_{P-1}F_{Q-1}(x)$ and such a case is always excluded from consideration in this paper. We assume all ${}_PF_Q$'s are irreducible.

Equation (1) can be given an interpretation for $P > Q + 1$ by means of the G-function

$$(5)\qquad \frac{\Gamma(b_Q)}{\Gamma(a_P)} G^{1,P}_{P,Q+1}\left(-x \,\middle|\, \begin{matrix} 1 - a_P \\ 0, 1 - b_Q \end{matrix}\right)$$

and (5) is (1) (or its analytic continuation) if $P \leqq Q + 1$. The G-function can be defined by a Mellin-Barnes contour integral.

For a treatment of the generalized hypergeometric function and the G-function, see [1].

We also assume that (5), wherever it occurs, is irreducible, i.e., that no a_i equals any b_j, $i = 1, 2, \cdots, P$, $j = 1, 2, \cdots, Q$.

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II. Introduction. The subject of the recursion relations satisfied by hypergeometric functions occupies a prominent place in the literature of special functions. The functions of this type for which recursion formulae have been given are usually special cases of the functions

$$U_n(\lambda) = \frac{(a_P)_n \lambda^n}{(b_Q)_n(\gamma + n)_n} \, {}_{P+1}F_{Q+1}\left(\begin{matrix} n + a_{P+1} \\ n + b_Q, 2n + \gamma + 1 \end{matrix} \middle| \lambda \right), \tag{6}$$

or of the polynomials

$$P_n(z) = \frac{1}{\Gamma(d_T)} \, {}_{R+2}F_T\left(\begin{matrix} -n, n + \gamma, c_R \\ d_T \end{matrix} \middle| z \right), \tag{7}$$

or

$$Q_n(z) = \frac{1}{\Gamma(d_T)} \, {}_{R+1}F_T\left(\begin{matrix} -n, c_R \\ d_T \end{matrix} \middle| z \right). \tag{8}$$

It can be shown that (6)–(8) obey linear recursion relationships of the form

$$\sum_{\nu=0}^{\rho} [k_\nu + x l_\nu] \Phi_{n+\nu} = 0, \tag{9}$$

where $x = 1/\lambda$ for (6), $x = z$ for (7) and (8), and $k_\nu = k_\nu(n)$, $l_\nu = l_\nu(n)$ depend on the particular function, but not on z or λ. Also, $k_0 = 1$, $l_0 = 0$, and ρ depends on the number of numerator and denominator parameters in the hypergeometric function: $\rho = \max [P + 1, Q + 2]$ for (6), $\rho = \max [T + 1, R + 2]$ for (7) and (8).

$U_n(\lambda)$ can be given an interpretation for $P > Q + 1$ by means of the G-function

$$\begin{aligned} U_n(\lambda) &= \frac{\Gamma(b_Q)}{\Gamma(a_P)} (-)^n \tau_n G^{1,P+1}_{P+1,Q+2}\left(-\lambda \middle| \begin{matrix} 1 - a_{P+1} \\ n, -n - \gamma, 1 - b_Q \end{matrix} \right), \\ \tau_n &= (2n + \gamma)\Gamma(n + \gamma)/\Gamma(n + \beta + 1), \end{aligned} \tag{10}$$

provided a_i is not 0 or a negative integer, $i = 1, 2, \cdots, P + 1$.

There exists a duality between the functions (7) and (10). For instance, we have, under a variety of conditions (see [2, Eq. (2.6)] and also related expansions in [3], [4]),

$$\begin{aligned} G^{P+R+1,1}_{Q+T+1,P+R+1}\left(-\frac{1}{\lambda z} \middle| \begin{matrix} 1, b_Q, d_T \\ c_R, a_{P+1} \end{matrix} \right) &= \frac{\Gamma(c_R)\Gamma(a_P)}{\Gamma(b_Q)} \sum_{n=0}^{\infty} (-)^n (n + 1)_\beta \\ &\quad \times U_n(\lambda) P_n(z), \end{aligned} \tag{11}$$

and if, in this multiplication formula, z is replaced by z/γ and $\gamma \to \infty$, a similar expansion in terms of $Q_n(z)$ results.

In fact, any function analytic at $z = 0$ can be expanded in a series of the polynomials P_n or Q_n, and Fields and Wimp studied such expansions from the standpoint of basic series in [6]. Linear combinations of P_n, Q_n also occur in classes of rational approximations to generalized hypergeometric functions, see [7] and the references given there.

For $R = 0$, $T = 1$, P_n is related to the Jacobi polynomial, as we have seen, and Q_n to the Laguerre polynomial. Here $\rho = 2$, and the recurrence formulae are classical. For $R = 0$, $T = 0$, P_n is the Bessel polynomial, whose recursion formula and other properties have recently been studied by a number of writers, see [8].

Recursion formulae for P_n for $R = 1$, $T = 2$ ($\rho = 3$) have been studied for various special values of the parameters, see [9]. For values of $\rho > 3$, i.e., larger values of R, T, no general results seem to exist in the literature, although general formulae for $\rho = 3$ have been derived but not published, [6].

When $P = 1$, $Q = 0$, then $\rho = 2$ and $U_n(\lambda)$ is related to the Jacobi function, $Q_n^{(\alpha,\beta)}$, whose recursion formula is given in [5]. No general formulae for larger values of P, Q seem to be known. However, for special values of γ and β, the recursion formula for $P = 2$, $Q = 0$ is given in [3], where it was also shown that $U_n(\lambda)$ could be computed by using (9) in the backward direction.

Since $U_n(\lambda)$ can often be computed by using (9) in the backward direction, and P_n and Q_n always by using (9) in the forward direction, it is quite desirable to have closed form expressions for l_ν, k_ν. It was previously doubted that such expressions existed, since the derivation of particular recursion formulae has hithertofore involved solving systems of algebraic equations whose complexity increases rapidly with P, Q, R and T.

In this paper, we determine closed form expressions for the coefficients in the recursion formula for $U_n(\lambda)$. These coefficients are terminating hypergeometric functions of unit argument. We show that $U_n(\lambda)$ satisfies one and only one recursion relation of type (9) of a certain order and none of a lower order. We next find a number of other solutions of (9), considered as a difference equation. It turns out that certain of these solutions are closely related to P_n, and by specialization of a certain parameter, we are able to determine the recursion formula for $P_n(z)$. Next, by taking a limit as $\gamma \to \infty$, we find the recursion formula for $Q_n(z)$.

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III. Results.

THEOREM 1. *Let P, Q, n be integers $\geqq 0$. Let β, γ, a_i, b_j, $i = 1, 2, \cdots, P$, $j = 1, 2, \cdots, Q$ be complex constants such that none of the quantities $\beta + 1$, a_i, b_j, γ are negative integers or zero. Let λ be a complex variable, finite and $\neq 0$, and let $a_i = \beta + 1$ for $i = P + 1$. Then the following statements are true*:

(1) *the functions $U_n(\lambda)$ as given by* (10) *satisfy the difference equation*

$$\sum_{\nu=0}^{\sigma}\left[A_\nu + \frac{B_\nu}{\lambda}\right]\Phi_{n+\nu}(\lambda) = 0\,, \qquad \sigma = \max\,[P + 1, Q + 2]\,, \tag{12}$$

where

$$\begin{aligned} A_\nu = {} & \frac{(-)^\nu(2n+\gamma)_\nu}{\nu!(n+\gamma)_\nu}(n+\beta+1) \\ & \times {}_{P+3}F_{P+2}\left(\begin{matrix} -\nu, 2n+\gamma+\nu, n+a_{P+1}+1 \\ 2n+\gamma+\sigma+1, n+a_{P+1} \end{matrix}\,\middle|\, 1\right), \end{aligned} \tag{13}$$

$$\begin{aligned} B_\nu = {} & \frac{(-)^\nu(2n+\gamma)_{\nu+1}(n+\beta+1)_\nu(n+b_Q)}{\Gamma(\nu)(n+\gamma)_\nu(n+a_{P+1})} \\ & \times {}_{Q+2}F_{Q+1}\left(\begin{matrix} 1-\nu, 2n+\gamma+\nu+1, n+b_Q+1 \\ 2n+\gamma+\sigma+1, n+b_Q \end{matrix}\,\middle|\, 1\right) \end{aligned} \tag{14}$$

($A_0 = 1, B_0 = B_\sigma = 0$);

(2) *other solutions of* (12) *are*

Case A. $\sigma = Q + 2$; $p < Q + 1$ *or* $P = Q + 1$, $|\arg(1 - \lambda)| < \pi$; (*here* $U_n(\lambda)$ *is given by* (6));

$$\begin{aligned}\psi_n(\lambda) &= \frac{(-)^{n(P+1)}\tau_n\lambda^{-n}}{\Gamma(b_Q - n - \gamma)\Gamma(n + \gamma + 1 - a_{P+1})\Gamma(1 - \gamma - 2n)} \\ &\quad \times {}_{P+1}F_{Q+1}\left(\begin{matrix} a_{P+1} - n - \gamma \\ b_Q - n - \gamma, 1 - \gamma - 2n \end{matrix} \middle| \lambda\right),\end{aligned} \tag{15}$$

$$\begin{aligned}\phi_n^{[h]}(\lambda) &= \frac{\tau_n}{\Gamma(2 - b_h - n)\Gamma(n + \gamma + 2 - b_h)\Gamma(1 + b_Q - b_h)} \\ &\quad \times {}_{P+1}F_{Q+1}\left(\begin{matrix} 1 + a_{P+1} - b_h \\ (1 + b_Q - b_h)^*, 2 - b_h - n, n + \gamma + 2 - b_h \end{matrix} \middle| \lambda\right),\end{aligned} \tag{16}$$

$h = 1, 2, \cdots, Q$;

Case B. $\sigma = P + 1$; $P > Q + 1$ *or* $P = Q + 1$, $|\arg(1 - 1/\lambda)| < \pi$;

$$\begin{aligned}\theta_n^{[h]}(\lambda) &= \frac{\tau_n(a_h)_n(-)^n}{\Gamma(n + \gamma + 1 - a_h)\Gamma(1 + a_h - a_{P+1})} \\ &\quad \times {}_{Q+2}F_P\left(\begin{matrix} n + a_h, -n - \gamma + a_h, 1 - b_Q + a_h \\ (1 + a_h - a_{P+1})^* \end{matrix} \middle| \frac{(-)^{Q+P+1}}{\lambda}\right),\end{aligned} \tag{17}$$

$h = 1, 2, \cdots, P + 1$;

(3) *none of the functions above satisfy any other difference equation of type* (12), *with* $A_0 = 1$, $B_0 = B_\sigma = 0$, *of order* $\leqq \sigma$.

Note. We assume U_n is not reducible for all n, i.e., no b_i equals any a_j or $\beta + 1$. However, for particular values of n, U_n may be reducible. Such will be the case if any $a_j = r + \gamma + 1$, $j = 1, 2, \cdots, P + 1$, r an integer $\geqq 0$.

Proof. First we note that

$$ {}_{M+2}F_{M+1}\left(\begin{matrix} -\nu, \nu + \mu, 1 + a_M \\ \mu + r, a_M \end{matrix} \middle| 1\right) = 0\,, \qquad \nu, r = 0, 1, 2, \cdots, \tag{18}$$

for $M < r \leqq \nu$, as can be seen by writing out the νth difference with respect to x of $\prod_{t=1}^{\nu-r}(x + r + \mu - 1 + t)\prod_{j=1}^{M}(x + a_j)$ at $x = 0$. This shows that, if (13) and (14) are true, then $A_\nu = 0$, $\nu > \sigma$ and $B_\nu = 0$, $\nu \geqq \sigma$, in particular, that $B_\sigma = 0$, as stated.

Next, we remark that if $P < Q + 1$, or $P = Q + 1$ and $|\arg(1 - \lambda)| < \pi$, then $U_n(\lambda)$ is precisely (6). If $P > Q + 1$ or $P = Q + 1$ and $|\arg(1 - 1/\lambda)| < \pi$, then $U_n(\lambda)$ is a sum of the functions $\theta_n^{[h]}(\lambda)$, $h = 1, 2, \cdots, P + 1$. See [10].

Let $P < Q + 1$ or $P = Q + 1$ and $|\lambda| < 1$. By substituting $U_n(\lambda)$ into the difference equation and equating to zero the coefficient of λ^{n+k}, we find that the theorem demands that

$$S_1(k) + S_2(k) \equiv 0\,, \tag{19}$$

where

$$S_1(k) = (n + b_Q + k)\sum_{\nu=0}^{\sigma}\frac{\tau_{n+\nu}A_\nu}{\Gamma(k - \nu + 1)\Gamma(2n + \nu + k + \gamma + 1)}\,, \tag{20}$$

$$S_2(k) = (n + a_{P+1} + k)\sum_{\nu=1}^{\sigma-1}\frac{\tau_{n+\nu}B_\nu}{\Gamma(k - \nu + 2)\Gamma(2n + \nu + k + \gamma + 2)}\,. \tag{21}$$

Now substitute the functions $\phi_n^{[h]}$ into (12) and equate to zero the coefficient of λ^k. The result is

$$S_1(k+1-n-b_h) + S_2(k+1-n-b_h) \equiv 0\,, \qquad h = 1, 2, \cdots, Q\,, \tag{22}$$

with the same value of σ as above.

Substituting $\psi_n(\lambda)$ into (12) and equating to zero the coefficient of λ^{-n+k}, we see we must have

$$S_1(k-2n-\gamma) + S_2(k-2n-\gamma) \equiv 0\,. \tag{23}$$

Finally, let $P > Q + 1$ or $P = Q + 1$ and $|\lambda| > 1$ and consider the functions $\theta_n^{[h]}(\lambda)$. Proceeding as above, we see that we must have

$$S_1(-k-a_h-n) + S_2(-k-a_h-n) \equiv 0\,, \qquad h = 1, 2, \cdots, P+1\,. \tag{24}$$

If (19) is multiplied by $\Gamma(k+1)\Gamma(2n+\sigma+k+\gamma+1)$ which is defined for all k in some right half-plane, then (19) becomes a polynomial in k, and we see that a necessary and sufficient condition for (19) to hold is that

$$(n+b_Q+k)f_1(k) + (n+a_{P+1}+k)f_2(k) \equiv 0\,, \tag{25}$$

$$f_1(k) = \sum_{\nu=0}^{\sigma} (-)^{\nu}(-k)_{\nu}(2n+k+\nu+\gamma+1)_{\sigma-\nu}\overline{A}_{\nu}\,, \tag{26}$$

$$f_2(k) = \sum_{\nu=1}^{\sigma-1} (-)^{\nu-1}(-k)_{\nu-1}(2n+k+\nu+\gamma+2)_{\sigma-\nu-1}\,\overline{B}_{\nu}\,, \tag{27}$$

where k is a generally complex-valued variable, and

$$\overline{A}_{\nu} = \tau_{n+\nu}A_{\nu}\,, \qquad \overline{B}_{\nu} = \tau_{n+\nu}B_{\nu}\,. \tag{28}$$

Thus, if $\overline{A}_{\nu}$, $\overline{B}_{\nu}$ can be chosen so that (25) holds, the functions U_n, ψ_n, $\phi_n^{[h]}$, $\theta_n^{[h]}$ will satisfy the difference equation whenever the series defining them converge, since (19)–(24) are all equivalent to (25)–(27).

We now discuss the quantity σ, which up till now has been unspecified.

Note that $f_1(k)$ is a polynomial in k of degree σ at most and, since no b_i equals any a_i or $\beta+1$, has zeros at $k = -n-a_i$, $i = 1, 2, \cdots, P+1$.

Or

$$f_1(k) \equiv (n+a_{P+1}+k)M_r(k)\,, \tag{29}$$

where $M_r(k)$ is a polynomial of degree r in k. Neither f_1 nor M_r can be identically zero, since

$$f_1(0) = (2n+\gamma+1)_{\sigma}\overline{A}_0\,. \tag{30}$$

Equation (29) shows that, for some integer m_1, $m_1 \geqq 0$, $\sigma - m_1 = P + r + 1$ or $\sigma \geqq P + 1$.

Likewise, f_2 is a polynomial of degree $\sigma - 2$ at most and

$$f_2(k) = (n+b_Q+k)N_s(k)\,, \tag{31}$$

where N_s is a polynomial of degree s in k. Setting $k = 0$ in (25) gives

$$\overline{B}_1 = -(n+b_Q)(2n+\gamma+1)_2\,\overline{A}_0/(n+a_{P+1}) \tag{32}$$

and clearly this is the only possible value of $\overline{B}_1$.

Furthermore,

$$f_2(0) = -(n + b_Q)(2n + \gamma + 1)_\sigma \overline{A}_0/(n + a_{P+1}) \tag{33}$$

so $N_s(k) \not\equiv 0, f_2(k) \not\equiv 0$; (31) shows that, for some integer $m_2 \geqq 0$, $\sigma - m_2 - 2 = Q + s$ or $\sigma \geqq Q + 2$.

Thus, the smallest possible value of σ is

$$\sigma = \max [P + 1, Q + 2] . \tag{34}$$

Assume σ has this value. We will show that $\overline{A}_\nu$, $\overline{B}_\nu$ (hence, A_ν, B_ν) are then uniquely determined by (25) and that $A_\sigma \not\equiv 0$, which means that no other recursion relationship of order $\leqq \sigma$ exists for any of the given functions, i.e., statement (3) of the theorem. (It is clear, however, that larger values of σ are possible, e.g., add to (12) the recursion relationship obtained by replacing n by $n + 1$ and the result is a recursion formula of order $\sigma + 1$.)

LEMMA 1. *Let the conditions of the theorem hold. Then* (25) *is true if and only if* $\overline{A}_\nu$, $\overline{B}_\nu$ *are such that*

$$f_1(k) \equiv (2n + \gamma + 1)_\sigma (n + a_{P+1} + k)\overline{A}_0/(n + a_{P+1}) , \tag{35}$$

$$f_2(k) \equiv -(2n + \gamma + 1)_\sigma (n + b_Q + k)\overline{A}_0/(n + a_{P+1}) . \tag{36}$$

If k is assigned σ distinct values in (35) and $\sigma - 2$ distinct values in (36), then $\overline{A}_\nu$, $\nu = 1, 2, \cdots, \sigma$ and $\overline{B}_\nu$, $\nu = 2, 3, \cdots, \sigma - 1$ are uniquely determined and so, by (28), are A_ν, B_ν. Also, $A_\sigma \not\equiv 0$.

Proof. First assume $P > Q + 1$, $\sigma = P + 1$. Then $f_1(k)$ is a polynomial of degree $P + 1$ at most. But since $f_1(k) \not\equiv 0$, (29) shows it must be exactly of degree $P + 1$, and

$$f_1(k) = K(n + a_{P+1} + k) . \tag{37}$$

Letting $k = 0$ and using (30) determines K, and when (35) is substituted into (25), (36) follows.

Let $P \leqq Q + 1$, $\sigma = Q + 2$; $f_2(k)$ is a polynomial in k of degree Q at most. As before, $f_2(k) \not\equiv 0$ and so

$$f_2(k) = K'(n + b_Q + k) . \tag{38}$$

Letting $k = 0$ and using (33) we find K' whence (36) follows. When (36) is substituted into (25), (35) results.

Now let σ distinct values k_i, $i = 1, 2, \cdots, \sigma$ be assigned to k in (35). The result is σ nonhomogeneous equations in the σ unknowns $\overline{A}_\nu$, $\nu = 1, 2, \cdots, \sigma$. Now this system has a unique solution which is independent of the values of k assigned.

Let V_R denote the alternate determinant

$$V_R(x_R) = |x_i^{\,j-1}|_{i,j=1,2,\cdots,R} = \prod_{m=2}^{R} \prod_{l=1}^{m-1} (x_m - x_{m-l}) . \tag{39}$$

Here and in what follows, τ_{ij} is the element in the ith row and jth column of the determinant $|\tau_{ij}|_{i,j=1,2,\cdots,R}$. The determinant of the system formed from (35) is

$$D = |(-)^{j-1}(1 - k_i)_{j-1}(2n + k_i + j + \gamma + 1)_{\sigma-j}|_{i,j=1,2,\cdots,\sigma} \tag{40}$$

which, by [11], is

$$D = KV_\sigma(k_\sigma) \tag{41}$$

and K is independent of the k_i's. To determine K, let $k_i = i$. The resulting determinant is triangular, and we find

$$D = V_\sigma(k_\sigma) \prod_{i=1}^{\sigma} (2n + 2i + \gamma + 1)_{\sigma-i} \tag{42}$$

so, under our hypotheses, $D \neq 0$. If the system is solved by Cramer's rule, it can be verified that $V_\sigma(k_\sigma)$ also factors out of each numerator determinant, leaving a quantity independent of the k_i's. Thus, $\bar{A}_\nu$ is uniquely determined by (35), and similarly one can show that $\bar{B}_\nu$ is uniquely determined by (36), with $\bar{B}_1$ given by (32). $\bar{A}_\sigma$, hence A_σ, can be found by putting $k = -\sigma - \gamma - 2n$ in (35), and the result is displayed in Theorem 2, Eq. (52). Under our hypothesis, $A_\sigma \not\equiv 0$.

It remains to prove that A_ν, B_ν are indeed given by (13) and (14). For this, we require two more lemmas.

LEMMA 2. *Let k, b and z be complex quantities, $b + k + 1 \neq 0, -1, -2, \cdots$, and s an integer $\geqq 0$. Then*

$$\sum_{\nu=0}^{s} \frac{(b + 2\nu)(-k)_\nu(b + z)_\nu}{(1 - z)_\nu(b + k + 1)_\nu} = \frac{z(k + b) + \dfrac{(-k)_{s+1}(b + z)_{s+1}}{(b + k + 1)_s(1 - z)_s}}{(z - k)}. \tag{43}$$

Remark. Since the left-hand side and the right-hand side of (43) are the same meromorphic function of z, they have the same residues at the simple poles $z = 1, 2, \cdots, s$ and possess the same limit as $z \to k$.

Proof. By induction on s.

LEMMA 3. *If*

$$f_k = \sum_{\nu=0}^{k} \frac{(-k)_\nu g_\nu}{(a + k)_\nu}, \qquad k = 0, 1, 2, \cdots, M \geqq 0, \tag{44}$$

then

$$g_\nu = \frac{(a + 2\nu - 1)}{\nu!} \sum_{s=0}^{\nu} \frac{(-\nu)_s(a + s)_\nu f_s}{s!(a + s + \nu - 1)} \tag{45}$$

provided $a \neq 0, -1, -2, \cdots$.

Proof. The determinant of the system is nonzero, so (44) has a unique solution. The lemma then results by substituting (45) in (44), interchanging the order of summation, and using Lemma 2 with $z = 0$.

Now, in (35) let $k = 0, 1, 2, \cdots, \sigma$. Then

$$f_k = \sum_{\nu=0}^{k} \frac{(-k)_\nu(-)^\nu \bar{A}_\nu}{(2n + \gamma + k + 1)_\nu} = \frac{(2n + \gamma + 1)_\sigma(n + a_{P+1} + k)\bar{A}_0}{(n + a_{P+1})(2n + \gamma + k + 1)_\sigma} \tag{46}$$

and this system is the form in Lemma 3 with $g_\nu = (-)^\nu \bar{A}_\nu$, $a = 2n + \gamma + 1$. Thus $\bar{A}_\nu$ and hence A_ν is easily found and the result is (13). $\bar{B}_\nu$ is similarly determined by applying Lemma 3 to (36).

The extension of the theorem to values λ such that $|\arg(1 - \lambda)| < \pi$ in Case A, $P = Q + 1$, or $|\arg(1 - 1/\lambda)| < \pi$ in Case B, $P = Q + 1$ is immediate by the permanence principle for functional equations [12].

The proof of Theorem 1 is complete.

Note that no restrictions on b_i enter in the proof of the theorem; the restriction that $b_i \neq 0, -1, -2, \cdots$, arises from the definition (6). In fact, by slightly modifying (12) (e.g., multiplying by $(n + a_{P+1})$) or the solutions of the difference equation (e.g., dividing $U_n(\lambda)$ by $\Gamma(b_Q)$), the theorem can be made valid for a_i, b_j negative integers. Also, Φ_n may be redefined so that the theorem will hold for all values of $\beta + 1$ and γ.

Now if no two of the quantities $[n, b_Q, -\gamma - n]$ differ by an integer or zero, all the solutions in Case A are distinct, and if no two of the quantities $[a_{P+1}]$ differ by an integer or zero, all the solutions in Case B are distinct. In fact, under these restrictions the functions in each group are linearly independent functions of λ, as is seen by comparing their behavior near $\lambda = 0$ or $\lambda = \infty$. This is not at all the same as asserting that the functions in either group are linearly independent as functions of n.

If $2n + \gamma$ is an integer, $\psi_n(\lambda)$ is proportional to $U_n(\lambda)$, while if two of the quantities $[b_Q]$ (or $[a_{P+1}]$) differ by an integer or zero, then two of the functions $[\phi_n^{[Q]}]$ (or $[\theta_n^{[P+1]}]$) are proportional. However, in any of these cases a distinct set of solutions can be constructed. For example, let $a_i = a_j + m$, $m = 0, 1, 2, 3, \cdots$. Then one forms an appropriate difference of the functions $\theta_n^{[i]}$, $\theta_n^{[j]}$ for $a_i = a_j + m + \epsilon$, divides by ϵ, and lets $\epsilon \to 0$. See [13] for the mechanics of this procedure.

We will subsequently need the following integral representations of (13) and (14).

LEMMA 4. *Let none of the quantities γ, a_i, $i = 1, 2, \cdots, P + 1$ be negative integers or zero. Then, for general σ, we have*

$$A_\nu = \frac{v_{n,\nu}}{2\pi i}\int_{\Gamma_\nu} \frac{\Gamma(2n + \gamma + \nu + z)\Gamma(-z)(n + a_{P+1} + z)dz}{\Gamma(2n + \gamma + \sigma + 1 + z)\Gamma(\nu + 1 - z)}, \tag{47}$$

$$B_\nu = \frac{v_{n,\nu}}{2\pi i}\int_{\Gamma_{\nu-1}} \frac{\Gamma(2n + \gamma + \nu + 1 + z)\Gamma(-z)(n + b_Q + z)dz}{\Gamma(2n + \gamma + \sigma + 1 + z)\Gamma(\nu - z)}, \tag{48}$$

$$v_{n,\nu} = \frac{(-)^{\nu+1}(2n + \gamma)_{\sigma+1}(n + \beta + 1)_\nu}{(n + \gamma)_\nu(a_{P+1} + n)} \tag{49}$$

and Γ_m denotes a simple closed path enclosing the points $z = 0, 1, 2, \cdots, m$ but no other singularities of the integrand.

Proof. By the residue theorem. Note that Γ_m is a feasible path since, were any of the poles of $\Gamma(2n + \gamma + \nu + z)$ (or $\Gamma(2n + \gamma + \nu + z + 1)$) to coincide with any of the poles of $\Gamma(-z)$, then γ would be zero or a negative integer.

We now give alternate representations of A_ν, B_ν which are useful when ν is larger than $[\sigma/2]$.

THEOREM 2. *Let none of the quantities γ, $\beta + 1$, a_i, $i = 1, 2, \cdots, P$ be negative integers or zero. Then*

$$\begin{aligned} A_\nu = {} & \frac{(-)^{\nu+P+1}(2n + \gamma)_{\sigma+1}(n + \beta + 1)_\nu(n + \gamma + \nu - a_{P+1})}{\Gamma(\sigma + 1 - \nu)(n + \gamma)_\nu(2n + \gamma + \nu)_{\nu+1}(n + a_{P+1})} \\ & \times {}_{P+3}F_{P+2}\left(\left.\begin{matrix} \nu - \sigma, 2n + \gamma + \nu, n + \gamma + \nu + 1 - a_{P+1} \\ 2n + \gamma + 2\nu + 1, n + \gamma + \nu - a_{P+1} \end{matrix}\right| 1\right), \end{aligned} \tag{50}$$

$$(51)\quad \begin{aligned} B_\nu = {} & \frac{(-)^{\nu+Q}(2n+\gamma)_{\sigma+1}(n+\beta+1)_\nu(n+\gamma+\nu+1-b_Q)}{\Gamma(\sigma-\nu)(n+\gamma)_\nu(2n+\gamma+\nu+1)_\nu(n+a_{P+1})} \\ & \times {}_{Q+2}F_{Q+1}\left(\begin{matrix}\nu+1-\sigma, 2n+\gamma+\nu+1, n+\gamma+\nu+2-b_Q \\ 2n+\gamma+2\nu+1, n+\gamma+\nu+1-b_Q\end{matrix}\middle| 1\right), \end{aligned}$$

and in particular

$$(52)\qquad A_\sigma = \frac{(-)^{\sigma+P+1}(2n+\gamma)_\sigma(n+\beta+1)_\sigma(n+\gamma+\sigma-a_{P+1})}{(n+\gamma)_\sigma(2n+\gamma+\sigma+1)_\sigma(n+a_{P+1})}.$$

Proof. We prove (50) only, since (51) follows similarly. Denote the integrand of (47) by $L_n(z)$. It has poles at the points $\delta_m = -2n-\gamma-m$, $m = \nu, \nu+1, \cdots, \sigma$ and γ_m, $m = 0, 1, 2, \cdots, \nu$. The integral around any large circle containing both $\{\gamma_m\}$ and $\{\delta_m\}$ is zero, since $L_n(z) = O\{z^{P-\sigma-1}\}$, $|z| \to \infty$, and is a rational function of z. If Δ_ν is any simple closed curve containing the points $\{\gamma_m\}$ but none of the points $\{\delta_m\}$, then

$$(53)\qquad \int_{\Gamma_\nu} = -\int_{\Delta_\nu}$$

and (50), and hence (52), follow immediately by the residue theorem. (Note the hypotheses separate the points $\{\gamma_m\}$ from $\{\delta_m\}$.)

Because of the form of the functions $\theta_n^{[h]}(\lambda)$, Theorems 1 and 2 enable us to give explicit recurrence formulae for the classes of hypergeometric polynomials studied in [4].

COROLLARY 1. *Let R and T be integers $\geqq 0$, $\tau = \max[T+1, R+2]$. Let γ, c_i, d_j, $i = 1, 2, \cdots, R$, $j = 1, 2, \cdots, T+1$, ($d_j = 1$ for $j = T+1$) be complex constants such that none of the quantities γ, $\gamma+1-d_j$, $j = 1, 2, \cdots, T$ are negative integers or zero. Then the hypergeometric polynomials $P_n(z)$, see (7), satisfy the recursion relationship*

$$(54)\qquad \sum_{\nu=0}^{\tau} [C_\nu + zD_\nu]P_{n-\nu}(z) = 0, \qquad n = \tau, \tau+1, \tau+2, \cdots,$$

where

$$(55)\quad \begin{aligned} C_\nu = {} & \frac{(-)^\nu(n+1-\nu)_\nu(1-\gamma-2n)_{2\nu}(n-\nu-1+d_{T+1})}{\nu!(n+\gamma-\nu)_\nu(\tau+1-\gamma-2n)_\nu(n+d_{T+1}-1)} \\ & \times {}_{T+3}F_{T+2}\left(\begin{matrix}-\nu, 2n+\gamma-\tau-\nu, n-\nu+d_{T+1} \\ 2n+\gamma+1-2\nu, n-\nu-1+d_{T+1}\end{matrix}\middle| 1\right) \end{aligned}$$

and

$$(56)\quad \begin{aligned} D_\nu = {} & \frac{(-)^{\nu+1}(n+1-\nu)_\nu(1-\gamma-2n)_{2\nu}(n-\nu+c_R)}{\Gamma(\nu)(n+\gamma-\nu)_\nu(1+\tau-\gamma-2n)_{\nu-1}(n+d_{T+1}-1)} \\ & \times {}_{R+2}F_{R+1}\left(\begin{matrix}1-\nu, 2n+\gamma+1-\tau-\nu, n+1-\nu+c_R \\ 2n+\gamma+1-2\nu, n-\nu+c_R\end{matrix}\middle| 1\right) \end{aligned}$$

and $D_0 = D_\tau = 0$.

Proof. In $\theta_n^{[P+1]}(\lambda)$ let $Q = R$, $P = T$, $a_j = \gamma+1-d_j$ ($d_{T+1} = 1$), $b_j = \gamma + 1 - c_j$, $\beta+1 = \gamma$, $z = (-)^{Q+P+1}/\lambda$, $\sigma = \tau$. Then (55) and (56) follow from Theorem 2 when the sums are turned around and n is replaced by $n - \tau$; since the polynomials are computed in the forward direction, this is the more useful form of the recursion relationship. Note that it is not necessary to assume $P > Q + 1$ in using Theorem 2. Since $\theta_n^{[P+1]}(\lambda)$ terminates, the recursion formula is valid for all P, Q. Also, alternate forms for C_ν, D_ν which are useful when $\nu > [\sigma/2]$ can be determined from Theorem 1.

COROLLARY 2. *Let R and T be integers* $\geqq 0$, $\tau = \max [T + 1, R + 2]$, *and let* c_i, d_j, $i = 1, 2, \cdots, R$, $j = 1, 2, \cdots, T + 1$ *be complex constants*, ($d_j = 1$ *for* $j = T + 1$). *Then the hypergeometric polynomials* $Q_n(z)$, *see* (8), *satisfy the recursion relationship*

$$\sum_{\nu=0}^{l_1} E_\nu Q_{n-\nu}(z) + z \sum_{\nu=1}^{l_2} F_\nu Q_{n-\nu}(z) = 0\,, \tag{57}$$

$l_1 = \min [\tau, T + 1]$, $l_2 = \min [\tau - 1, R + 1]$, $n = \tau + \delta, \tau + \delta + 1, \tau + \delta + 2, \cdots$, $\delta = 0$ or -1, *where*

$$E_\nu = \frac{(n + 1 - \nu)_\nu (n - \nu - 1 + d_{T+1})}{\nu!(n + d_{T+1} - 1)}\, {}_{T+2}F_{T+1}\left(\begin{matrix} -\nu, n - \nu + d_{T+1} \\ n - \nu - 1 + d_{T+1} \end{matrix}\,\middle|\, 1\right), \tag{58}$$

$$F_\nu = \frac{(n + 1 - \nu)_\nu (n - \nu + c_R)}{\Gamma(\nu)(n + d_{T+1} - 1)}\, {}_{R+1}F_R\left(\begin{matrix} 1 - \nu, n + 1 - \nu + c_R \\ n - \nu + c_R \end{matrix}\,\middle|\, 1\right). \tag{59}$$

Proof. Let

$$Q_n^{(\gamma)}(z) = P_n(z/\gamma)\,. \tag{60}$$

Then

$$\lim_{\gamma \to \infty} Q_n^{(\gamma)}(z) = Q_n(z)\,. \tag{61}$$

If we form the difference equation for $Q_n^{(\gamma)}(z)$ we see we must have

$$\lim_{\gamma\to\infty} C_\nu = E_\nu\,, \qquad \lim_{\gamma\to\infty} \gamma^{-1} D_\nu = E_\nu\,. \tag{62}$$

Using (55), (56) to take the limits term by term gives (58) and (59).

Note that E_ν vanishes for $\nu > T + 1$ and F_ν for $\nu > R + 1$ since they may be expressed as the νth difference of $(n + d_{T+1} - 1 - \nu + x)$ or the $(\nu - 1)$th difference of $(n + c_R - \nu + x)$ respectively evaluated at $x = 0$.

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Jacobi Polynomial Expansions of a Generalized Hypergeometric Function over a Semi-Infinite Ray

By Y. L. Luke and J. Wimp

1. Introduction. Suppose $f(x)$ is continuous and has a piecewise continuous derivative for $0 \leqq x/\lambda \leqq 1$. Then $f(x)$ may be expanded into a uniformly convergent series of shifted Jacobi polynomials in the form

$$f(x) = \sum_{n=0}^{\infty} a_n(\lambda) R_n^{(\alpha,\beta)}(x/\lambda), \tag{1.1}$$

$$\epsilon \leqq x/\lambda \leqq 1 - \epsilon, \quad \epsilon > 0; \quad \alpha > -1, \quad \beta > -1,$$

where $R_n^{(\alpha,\beta)}(x) = P_n^{(\alpha,\beta)}(2x - 1)$ and the latter is the usual notation for the Jacobi polynomial [1, Ch. 10]. Various techniques are available for the determination of the coefficients $a_n(\lambda)$. In this connection, see, for example, the references [2, 3, 4, 5, 6, 7].

Suppose that $f(x)$ satisfies the above conditions for $1 \leqq x/\lambda \leqq \infty$ where $|\arg \lambda| < \varphi$. Then we may write

$$f(x) = \sum_{n=0}^{\infty} b_n(\lambda) R_n^{(\alpha,\beta)}(\lambda/x), \tag{1.2}$$

$$\epsilon \leqq \lambda/x \leqq 1 - \epsilon, \quad \epsilon > 0; \quad \alpha > -1, \quad \beta > -1.$$

If $f(x)$ has an asymptotic expansion of the form

$$f(x) \sim \sum_{n=0}^{\infty} c_n x^{-n}, \qquad x \to \infty, \qquad |\arg x| < \varphi, \tag{1.3}$$

then (1.2) may be interpreted as a summability process which converts the generally divergent expansion (1.3) into a convergent expansion. If $f(x)$ in (1.3) is of hypergeometric type,* then the coefficients $b_n(\lambda)$ may be found formally at least using the procedures [5, 6]. These yield for $b_n(\lambda)$ an asymptotic series in λ which is also of hypergeometric type. The asymptotic representation for $b_n(\lambda)$ in general is not suitable for computation. We are confronted with two problems: one is the interpretation of the asymptotic series for $b_n(\lambda)$, and the other is the computation of $b_n(\lambda)$.

In this paper, we show how both problems can be solved for a confluent hypergeometric function. Actually we derive a representation for $b_n(\lambda)$ when $f(x)$ is the G-function, which includes the confluent hypergeometric function as a special case. Our computational scheme for $b_n(\lambda)$ is exhibited only when $f(x)$ is a confluent hypergeometric function, although the ideas involved can be extended to cover other special cases of the G-function as well.

In Section II, we prove an expansion theorem of the form (1.2) when $f(x)$ is the

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* For the definition and properties of generalized hypergeometric series including the G-function as well as other notations used in this paper, see [1, Chs. 4, 5, 6].

G-function and show how both convergent and asymptotic representations for $b_n(\lambda)$ may be derived. These results are specialized in Section III for the case when $f(x)$ is a confluent hypergeometric function, and in Section IV it is shown how $b_n(\lambda)$ may be computed by a recursion scheme. Finally, in Section V, we tabulate coefficients for the cases where $R_n^{(\alpha,\beta)}(x)$ is the shifted Chebyshev polynomial and $f(x)$ is the error function, the exponential, sine and cosine integrals, and the Bessel functions $K_0(x)$ and $K_1(x)$.

2. Expansion of the G-Function. The G-function is given by the Mellin-Barnes integral

$$G_{p,q}^{m,k}\left(\lambda x \,\middle|\, {}^{a_p}_{b_q}\right) = \frac{1}{2\pi i}\int_L \frac{\prod_{j=1}^{m}\Gamma(b_j - s)\prod_{j=1}^{k}\Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + s)\prod_{j=k+1}^{p}\Gamma(a_j - s)}(\lambda x)^s\,ds, \tag{2.1}$$

where an empty product is interpreted as 1, $0 \leqq m \leqq q$, $0 \leqq n \leqq p$ and the parameters are such that no pole of $\Gamma(b_j - s)$, $j = 1, 2, \cdots, m$ coincides with any pole of $\Gamma(1 - a_h + s)$, $h = 1, 2, \cdots, k$. We assume x is real and the path L runs parallel to the imaginary axis and is indented so that the poles of $\Gamma(b_j - s)$, $j = 1, 2, \cdots, m$, are to the right, and all the poles of $\Gamma(1 - a_h + s)$, $h = 1, 2, \cdots, k$, to the left of L. The integral converges if $p + q < 2(m + k)$ and $|\arg\lambda| < (m + k - p/2 - q/2)\pi$. For a treatment of the G-function, see [1, Ch. 5].

Now from [1, 10.20(3)] we have the expansion

$$\begin{aligned} x^s &= \Gamma(\beta - s + 1)\Gamma(1 - s) \\ &\times \sum_{n=0}^{\infty} \frac{(2n + \alpha + \beta + 1)(n + \beta + 1)_\alpha}{\Gamma(n + \alpha + \beta + 2 - s)\Gamma(1 - s - n)} R_n^{(\alpha,\beta)}(1/x), \quad 1 < x < \infty, \end{aligned} \tag{2.2}$$

uniformly for $\mathrm{Re}\,(s) \leqq \mu - \delta$, $\delta > 0$, $\mu = \min(\beta + 1, \beta/2 + \frac{3}{4})$, $\alpha > -1$, $\beta > -1$. Put (2.2) in (2.1) and integrate along the path from $\mu - \delta - i\infty$ to $\mu - \delta + i\infty$. We then get

THEOREM I. *Let*

1. α, β *and* x *be real*, $\alpha > -1$, $\beta > -1$, $1 < x < \infty$.

Let a real positive δ *exist such that*

2. (a) $\mathrm{Re}\,(a_j - 1) < \mu - \delta$, $j = 1, 2, \cdots k$; (b) $\mathrm{Re}\,(b_j) > \mu - \delta$, $j = 1, 2, \cdots m$, $\mu - \delta < 1$, $\mu = \min(\beta + 1, \beta/2 + \frac{3}{4})$.

3. $p + q < 2(m + k)$, $|\arg\lambda| < (m + k - p/2 - q/2)\pi$, $\lambda \neq 0$, $0 \leqq m \leqq q$, $0 \leqq k \leqq p$.

Then

$$\begin{aligned} G_{p,q}^{m,k}\left(\lambda x_b^a \,\middle|\, {}^{p}_{q}\right) &= \sum_{n=0}^{\infty}(2n + \alpha + \beta + 1)(n + \beta + 1)_\alpha \\ &\times G_{p+2,q+2}^{m+2,k}\left(\lambda \,\middle|\, {}^{a_p,1-n,n+\alpha+\beta+2}_{1,\beta+1,b_q}\right) R_n^{(\alpha,\beta)}(1/x). \end{aligned} \tag{2.3}$$

Remark. Assumptions 2 above insure the separation of poles and specify the regions in which they lie according to the remarks surrounding (2.1). Notice, however, that poles of $\Gamma(b_j - s)$ may lie to the left of the contour. They may be excluded by indentations since they lie in a region where the series for x^s converges uniformly, provided they do not coincide with any of the poles of $\Gamma(1 - a_h + s)$. Hence, we may replace 2(b) by the weaker but more complicated condition

2(b)* $1 + \delta_{j-2} - a_h \neq 0, -1, -2, \cdots,$

$$j = 1, 2, \cdots m + 2, h = 1, 2, \cdots k, \delta_{-2} = 1, \delta_{-1} = \beta + 1, \delta_{j-2} = b_j, j > 1.$$

Notice from the definition of the G-function that

$$G_{p+2,q+2}^{m+2,k}\left(\lambda \,\middle|\, {}^{a_p, 1-n, n+\alpha+\beta+2}_{1, \beta+1, b_q}\right) = (-)^n G_{p+2,q+2}^{m+1,k+1}\left(\lambda \,\middle|\, {}^{1-n, a_p, n+\alpha+\beta+2}_{\beta+1, b_q, 1}\right). \tag{2.4}$$

If $|\arg \lambda| < \frac{1}{2}(p - q + 1)\pi$, an asymptotic representation for the coefficients of $R_n^{(\alpha,\beta)}(1/x)$ in (2.3) follows by application of a result in [1, 5.3(6)]. An ascending series representation follows when [1, 5.3(5)] is applied to the right-hand side of (2.4).

3. Expansion of a Confluent Hypergeometric Function. We consider the function [1, Ch. 6],

$$(\lambda x)^a \psi(a, c \mid \lambda x) = \{\Gamma(a)\Gamma(\sigma)\}^{-1} G_{1,2}^{2,1}\left(\lambda x \,\middle|\, {}^{1}_{a,\sigma}\right), \quad \sigma = a + 1 - c. \tag{3.1}$$

Also, denote by $T_n^*(x)$ the shifted Chebyshev polynomial

$$T_n^*(x) = T_n(2x - 1) = \frac{n!}{(\frac{1}{2})_n} R_n^{(-1/2,-1/2)}(x). \tag{3.2}$$

From Theorem I, we get

THEOREM II. *Let*

1. $1 \leqq x \leqq \infty$;
2. $\sigma \neq 0, -1, -2, \cdots; a \neq 0, -1, -2, \cdots;$
3. $|\arg \lambda| < 3\pi/2, \lambda \neq 0$.

Then

$$(\lambda x)^a \psi(a, c \mid \lambda x) = \sum_{n=0}^{\infty} C_n(\lambda) T_n^*(1/x), \tag{3.3}$$

where

$$C_n(\lambda) = \frac{\epsilon_n}{\pi^{1/2}\Gamma(a)\Gamma(\sigma)} G_{3,4}^{4,1}\left(\lambda \,\middle|\, {}^{1, 1-n, n+1}_{1, 1/2, a, \sigma}\right), \quad \epsilon_0 = 1, \epsilon_n = 2, n > 0, \tag{3.4}$$

or

$$C_n(\lambda) = \frac{\epsilon_n(-)^n}{\pi^{1/2}\Gamma(a)\Gamma(\sigma)} G_{2,3}^{3,1}\left(\lambda \,\middle|\, {}^{1-n, n+1}_{1/2, a, \sigma}\right). \tag{3.5}$$

Also, if none of the quantities $\frac{1}{2}$, a and σ differ by an integer

$$\begin{aligned} C_n(\lambda) = \frac{\epsilon_n(-)^n}{\pi^{1/2}} \Bigg\{ & (a)_{-1/2}(\sigma)_{-1/2}\lambda^{1/2} {}_2F_2\left({}^{n+1/2, -n+1/2}_{3/2-a, 3/2-\sigma} \,\middle|\, \lambda\right) \\ & + \frac{\Gamma(\frac{1}{2} - a)(a)_n(\sigma)_{-a}}{\Gamma(n + 1 - a)} \lambda^a {}_2F_2\left({}^{n+a, -n+a}_{a+1/2, a-\sigma+1} \,\middle|\, \lambda\right) \\ & + \frac{\Gamma(\frac{1}{2} - \sigma)(\sigma)_n(a)_{-\sigma}}{\Gamma(n - \sigma + 1)} \lambda^\sigma {}_2F_2\left({}^{n+\sigma, -n+\sigma}_{\sigma+1/2, \sigma-a+1} \,\middle|\, \lambda\right) \Bigg\}, \end{aligned} \tag{3.6}$$

and

$$(3.7)\quad C_n(\lambda) \sim \frac{\epsilon_n(-)^n(a)_n(\sigma)_n}{n!(4\lambda)^n}\, {}_3F_1\left(\begin{matrix} n+1/2, n+a, n+\sigma \\ 2n+1 \end{matrix} \middle| -\frac{1}{\lambda}\right), \quad |\lambda| \to \infty, \quad |\arg \lambda| < \pi.$$

Remark. Condition 1 of Theorem I is conservative. By an appeal to the convergence properties of expansions in Chebyshev polynomials [7], the range of x may be extended to give condition 1 above.

Since (3.3) converges,

$$(3.8)\qquad \lim_{n\to\infty} C_n(\lambda) = 0.$$

For later use, we record the fact that

$$(3.9)\qquad \lim_{x\to\infty} (\lambda x)^a \psi(a, c \mid \lambda x) = 1, \qquad |\arg \lambda| < 3\pi/2.$$

4. Calculation of the Coefficients $C_n(\lambda)$. Let

$$(4.1)\qquad \varphi_{1,n}(\lambda) = \frac{(-)^n}{\epsilon_n} C_n(\lambda).$$

Following the method developed in [8], we can show from the representation (3.7) that $\varphi_{1,n}(\lambda)$ satisfies the recursion relation

$$(4.2)\quad \varphi_n(\lambda) + (A_n + B_n\lambda)\varphi_{n+1}(\lambda) + (C_n + D_n\lambda)\varphi_{n+2}(\lambda) + E_n\varphi_{n+3}(\lambda) = 0,$$

where

$$(4.3)\qquad \begin{aligned} A_n &= (2n+2)\left[1 - \frac{(n+\frac{3}{2})(n+a+1)(n+\sigma+1)}{(n+2)(n+a)(n+\sigma)}\right], \\ B_n &= D_n = -4(n+1)/(n+a)(n+\sigma), \\ C_n &= -1 + [2(n+1)(2n+3)/(n+a)(n+\sigma)], \\ E_n &= -(n+1)(n-a+3)(n-\sigma+3)/(n+2)(n+a)(n+\sigma). \end{aligned}$$

We prove that the coefficients may be readily evaluated using (4.2) in the backward direction. This backward recursion technique has been treated by many authors [9], [10], [11], [12], [13]. The idea is as follows.

For fixed λ, arbitrary η and ν sufficiently large set

$$(4.4)\qquad \varphi_\nu^{(\nu)}(\lambda) = \varphi_{\nu-1}^{(\nu)}(\lambda) = 0,$$

$$(4.5)\qquad \varphi_{\nu-2}^{(\nu)}(\lambda) = \eta.$$

The sequence $\varphi_{\nu-3}^{(\nu)}(\lambda), \cdots, \varphi_n^{(\nu)}(\lambda), \cdots \varphi_1^{(\nu)}(\lambda), \varphi_0^{(\nu)}(\lambda)$ is generated from (4.2). Using (3.9) and

$$(4.6)\qquad T_n^*(0) = (-)^n$$

in (3.3) we would expect that if

$$(4.7)\qquad \omega_\nu = \sum_{n=0}^{\nu-2} \epsilon_n \varphi_n^{(\nu)}(\lambda),$$

then

$$C_n(\lambda) \sim (-)^n \epsilon_n \varphi_n^{(\nu)}(\lambda)/\omega_\nu, \tag{4.8}$$

with increasing accuracy as $\nu \to \infty$. In fact if we define

$$\varphi_{1,n}^{(\nu)}(\lambda) = \varphi_{1,0}(\lambda)\varphi_n^{(\nu)}(\lambda)/\varphi_0^{(\nu)}(\lambda), \tag{4.9}$$

we have:

THEOREM III. *Let* $|\arg \lambda| < \pi$, $\lambda \neq 0$, *and neither* a *nor* σ *be a negative integer or zero. Then*

$$\lim_{\nu\to\infty} \varphi_{1,n}^{(\nu)}(\lambda) = \varphi_{1,n}(\lambda). \tag{4.10}$$

Proof. Denote by $\varphi_{1,n}(\lambda)$, $\varphi_{2,n}(\lambda)$ and $\varphi_{3,n}(\lambda)$ the three linearly independent solutions of (4.2); $\varphi_{1,n}(\lambda)$ is the solution we wish to calculate. We may write*

$$\varphi_n^{(\nu)} = \xi_1^{(\nu)}\varphi_{1,n} + \xi_2^{(\nu)}\varphi_{2,n} + \xi_3^{(\nu)}\varphi_{3,n}, \qquad n < \nu - 2, \tag{4.11}$$

and the conditions (4.4) and (4.5) give

$$0 = \xi_1^{(\nu)}\varphi_{1,\nu} + \xi_2^{(\nu)}\varphi_{2,\nu} + \xi_3^{(\nu)}\varphi_{3,\nu}, \tag{4.12}$$

$$0 = \xi_1^{(\nu)}\varphi_{1,\nu-1} + \xi_2^{(\nu)}\varphi_{2,\nu-1} + \xi_3^{(\nu)}\varphi_{3,\nu-1}, \tag{4.13}$$

$$\eta = \xi_1^{(\nu)}\varphi_{1,\nu-2} + \xi_2^{(\nu)}\varphi_{2,\nu-2} + \xi_3^{(\nu)}\varphi_{3,\nu-2}, \tag{4.14}$$

where $\xi_1^{(\nu)}$, $\xi_2^{(\nu)}$ and $\xi_3^{(\nu)}$ are independent of n.

$$\xi_2^{(\nu)}/\xi_1^{(\nu)} = \gamma_\nu, \qquad \xi_3^{(\nu)}/\xi_1^{(\nu)} = \delta_\nu, \tag{4.15}$$

$$\gamma_\nu = [-\varphi_{1,\nu}\varphi_{3,\nu-1} + \varphi_{1,\nu-1}\varphi_{3,\nu}]/\tau_\nu, \tag{4.16}$$

$$\delta_\nu = [-\varphi_{2,\nu}\varphi_{1,\nu-1} + \varphi_{1,\nu}\varphi_{2,\nu-1}]/\tau_\nu, \tag{4.17}$$

$$\tau_\nu = [\varphi_{2,\nu}\varphi_{3,\nu-1} - \varphi_{3,\nu}\varphi_{2,\nu-1}]. \tag{4.18}$$

Thus

$$\varphi_{1,n}^{(\nu)} = \frac{\varphi_{1,n}\{1 + (\gamma_\nu\varphi_{2,n}/\varphi_{1,n}) + (\delta_\nu\varphi_{3,n}/\varphi_{1,n})\}}{\{1 + (\gamma_\nu\varphi_{2,n}/\varphi_{1,0}) + (\delta_\nu\varphi_{3,0}/\varphi_{1,0})\}}. \tag{4.19}$$

We will show that

$$\lim_{\nu\to\infty} \gamma_\nu = \lim_{\nu\to\infty} \delta_\nu = 0. \tag{4.20}$$

Equation (3.8) gives

$$\lim_{\nu\to\infty} \varphi_{1,\nu} = 0. \tag{4.21}$$

It may be directly verified that

$$\varphi_{2,n} = {}_2F_2\left({n+1/2,-n+1/2 \atop 3/2-a,3/2-\sigma} \,\middle|\, \lambda\right), \tag{4.22}$$

is also a solution of (4.2). From [14] we have

$$\varphi_{2,n} = C_1 n^{2/3[a+\sigma-2]} \exp\left[\tfrac{3}{2}n^{2/3}\lambda^{1/3}\right]\left[1 + O\left(\frac{1}{n}\right)\right], \qquad |\arg\lambda| < \pi, \tag{4.23}$$

* Henceforth we write, $\xi_1^{(\nu)}(\lambda) = \xi_1^{(\nu)}$, $\varphi_{1,n}(\lambda) = \varphi_{1,n}$, etc.

TABLE I
Coefficients for the Series

$$-Ei(-x) = \int_x^\infty \frac{e^{-t}}{t}\,dt = \frac{e^{-x}}{x}\sum_{n=0}^{\infty} A_n T_n^*\left(\frac{4}{x}\right), \qquad 4 \leqq x \leqq \infty,$$

$$Erfc(x) = \int_x^\infty e^{-t^2}\,dt = \frac{e^{-x^2}}{2x}\sum_{n=0}^{\infty} B_n T_{2n}\left(\frac{2}{x}\right), \qquad 2 \leqq x \leqq \infty.$$

n	A_n	B_n	n	A_n	B_n
0	0.90535 40999 62349 15873 (00)	0.94960 80415 75614 24493 (00)	19	−0.47291 68 (−13)	−0.83782 0 (−14)
1	−0.86481 17855 25987 1490 (−01)	−0.47027 51541 55887 6766 (−01)	20	0.14637 62 (−13)	0.25450 2 (−14)
2	0.72241 01543 74659 475 (−02)	0.30329 41457 65811 336 (−02)	21	−0.46173 9 (−14)	−0.78866 (−15)
3	−0.80975 59457 55738 62 (−03)	−0.29024 22664 89234 09 (−03)	22	0.14827 1 (−14)	0.24900 (−15)
4	0.10999 13443 26613 89 (−03)	0.35220 95068 75272 1 (−04)	23	−0.48417 (−15)	−0.8001 (−16)
5	−0.17173 32998 93776 7 (−04)	−0.50403 49016 87590 (−05)	24	0.16062 (−15)	0.2614 (−16)
6	0.29856 27514 47928 (−05)	0.81642 08349 9637 (−06)	25	−0.5409 (−16)	−0.867 (−17)
7	−0.56596 49145 7719 (−06)	−0.14583 31253 5019 (−06)	26	0.1847 (−16)	0.292 (−17)
8	0.11526 80839 7141 (−06)	0.28219 97993 581 (−07)	27	−0.640 (−17)	−0.100 (−17)
9	−0.24950 30440 269 (−07)	−0.58403 61925 03 (−08)	28	0.224 (−17)	0.35 (−18)
10	0.56923 24201 83 (−08)	0.12803 21362 01 (−08)	29	−0.80 (−18)	−0.12 (−18)
11	−0.13599 57664 81 (−08)	−0.29508 80097 7 (−09)	30	0.29 (−18)	0.4 (−19)
12	0.33846 62888 8 (−09)	0.71082 35787 (−10)	31	−0.10 (−18)	−0.2 (−19)
13	−0.87378 53904 (−10)	−0.17809 90369 (−10)	32	0.4 (−19)	0.1 (−19)
14	0.23315 88663 (−10)	0.46230 4869 (−11)	33	−0.1 (−19)	
15	−0.64114 8105 (−11)	−0.12391 4208 (−11)			
16	0.18122 4698 (−11)	0.34199 361 (−12)			
17	−0.52538 318 (−12)	−0.96955 55 (−13)			
18	0.15592 183 (−12)	0.28176 05 (−13)			

TABLE II

Coefficients for the Series

$$K_0(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x} \sum_{n=0}^{\infty} A_n T_n^*\left(\frac{2}{x}\right), \qquad 2 \leqq x \leqq \infty,$$

$$K_1(x) = \sqrt{\frac{\pi}{2x}}\, e^{-x} \sum_{n=0}^{\infty} B_n T_n^*\left(\frac{2}{x}\right), \qquad 2 \leqq x \leqq \infty.$$

n	A_n		B_n		n	A_n		B_n	
0	0.97354 00764 30036 78069	(+00)	1.08537 28165 51726 12040	(+00)	16	0.64093 20	(−13)	−0.71623 74	(−13)
1	−0.25091 95450 33808 0930	(−01)	0.82919 14491 55864 9624	(−01)	17	−0.17772 98	(−13)	0.19764 83	(−13)
2	0.12525 86114 67721 930	(−02)	−0.22802 07949 89514 525	(−02)	18	0.50586 5	(−14)	−0.56010 2	(−14)
3	−0.10252 45722 44517 42	(−03)	0.15575 94482 17418 05	(−03)	19	−0.14749 6	(−14)	0.16266 5	(−14)
4	0.11130 34099 23675 6	(−04)	−0.15448 62470 24490 3	(−04)	20	0.43980	(−15)	−0.48328	(−15)
5	−0.14615 29450 74297	(−05)	0.19200 97185 68380	(−05)	21	−0.13390	(−15)	0.14666	(−15)
6	0.22075 97885 5320	(−06)	−0.27941 60297 7437	(−06)	22	0.4157	(−16)	−0.4540	(−16)
7	−0.37185 32935 143	(−07)	0.45807 22385 967	(−07)	23	−0.1315	(−16)	0.1431	(−16)
8	0.68410 89366 29	(−08)	−0.82547 24141 10	(−08)	24	0.423	(−17)	−0.459	(−17)
9	−0.13544 36576 47	(−08)	0.16077 77088 92	(−08)	25	−0.138	(−17)	0.150	(−17)
10	0.28543 50058 7	(−09)	−0.33434 19366 8	(−09)	26	0.46	(−18)	−0.50	(−18)
11	−0.63491 57811	(−10)	0.73551 51365	(−10)	27	−0.15	(−18)	0.17	(−18)
12	0.14808 33144	(−10)	−0.16994 68453	(−10)	28	0.5	(−19)	−0.6	(−19)
13	−0.36021 2795	(−11)	0.41008 3914	(−11)	29	−0.2	(−19)	0.2	(−19)
14	0.90986 015	(−12)	−0.10286 1200	(−11)	30	0.1	(−19)	−0.1	(−19)
15	−0.23777 733	(−12)	0.26716 524	(−12)					

TABLE III

Coefficients for the Series

$$Si(x) = \int_x^\infty \frac{\sin t}{t}\,dt = \sum_{n=0}^{\infty} \left\{ \frac{A_n \cos x}{x} + \frac{B_n \sin x}{x} \right\} T_n^*\left(\frac{4}{x}\right), \qquad 4 \leqq x \leqq \infty,$$

$$Ci(x) = \int_x^\infty \frac{\cos t}{t}\,dt = \sum_{n=0}^{\infty} \left\{ \frac{B_n \cos x}{x} - \frac{A_n \sin x}{x} \right\} T_n^*\left(\frac{4}{x}\right), \qquad 4 \leqq x \leqq \infty.$$

n	A_n	B_n	n	A_n	B_n
0	0.96578 82803 51851 83021 (00)	0.10728 86713 38433 09526 (00)	21	0.17469 921 (−12)	−0.81934 01 (−13)
1	−0.43060 83777 85967 3425 (−01)	0.99693 56055 36349 5732 (−01)	22	−0.38470 01 (−13)	0.60587 82 (−13)
2	−0.73143 71174 81046 083 (−02)	−0.81628 39500 94241 970 (−02)	23	0.20193 (−15)	−0.27093 29 (−13)
3	0.14705 23578 98680 654 (−02)	−0.29696 08630 56773 29 (−03)	24	0.53597 7 (−14)	0.88808 3 (−14)
4	−0.98657 68573 27002 1 (−04)	0.22891 94548 45829 18 (−03)	25	−0.35606 0 (−14)	−0.18751 4 (−14)
5	−0.22743 20220 46550 8 (−04)	−0.41721 77635 53092 6 (−04)	26	0.15788 2 (−14)	−0.8110 (−16)
6	0.98240 25732 25254 (−05)	0.21254 28930 87307 (−05)	27	−0.52547 (−15)	0.34480 (−15)
7	−0.18973 43014 87133 (−05)	0.13157 50436 91368 (−05)	28	0.11378 (−15)	−0.22564 (−15)
8	0.10063 43594 1558 (−06)	−0.55848 57495 6974 (−06)	29	0.512 (−17)	0.10258 (−15)
9	0.80819 36482 224 (−07)	0.12353 72625 6029 (−06)	30	−0.2244 (−16)	−0.3571 (−16)
10	−0.38976 28287 529 (−07)	−0.10318 72179 187 (−07)	31	0.1532 (−16)	0.845 (−17)
11	0.10335 65032 550 (−07)	−0.50159 03675 67 (−08)	32	−0.732 (−17)	−0.3 (−19)
12	−0.14104 34487 59 (−08)	0.30915 99889 01 (−08)	33	0.273 (−17)	−0.146 (−17)
13	−0.25232 07840 0 (−09)	−0.10080 57370 10 (−08)	34	−0.73 (−18)	0.110 (−17)
14	0.25699 83132 6 (−09)	0.20289 59643 1 (−09)	35	0.6 (−19)	−0.56 (−18)
15	−0.10597 88925 4 (−09)	−0.27237 6669 (−11)	36	0.9 (−19)	0.23 (−18)
16	0.28970 03157 (−10)	−0.19967 52281 (−10)	37	−0.8 (−19)	−0.7 (−19)
17	−0.41023 1426 (−11)	0.11219 38506 (−10)	38	0.5 (−19)	0.1 (−19)
18	−0.10437 6937 (−11)	−0.40081 1186 (−11)	39	−0.2 (−19)	0.0 (−20)
19	0.10994 1845 (−11)	0.96702 841 (−12)	40	0.1 (−19)	−0.1 (−19)
20	−0.52214 239 (−12)	−0.71861 28 (−13)			

where C_1 is independent of n. The third linearly independent solution of (4.2) is the $L_{2,2}(-\lambda)$ term appearing in [15, 1.3.3(15)] which arises in the asymptotic expansion of (4.22) for large λ. A limit process, explained in [15, 1.3.4] is used to obtain $\varphi_{3,n}$, but our discussion here is necessarily brief. We need only the estimate

$$\varphi_{3,n} = \frac{C_2\Gamma(n+a-1)\Gamma(n+\sigma-1)}{(4\lambda)^n n!}\left[1+O\left(\frac{1}{n^2}\right)\right], \tag{4.24}$$

where C_2 is independent of n. Thus

$$\lim_{\nu\to\infty} |\varphi_{2,\nu}| = \lim_{\nu\to\infty} |\varphi_{3,\nu}| = \infty. \tag{4.25}$$

Also, from (4.23) and (4.24), we have

$$\tau_\nu = -\varphi_{2,\nu}\varphi_{3,\nu}\left[1+O\left(\frac{1}{\nu}\right)\right]. \tag{4.26}$$

Hence (4.20) is easily shown and the statement (4.10) follows from (4.19).

5. Tables. Tables I–III contain coefficients to 20 D for the expansions of several important cases of the confluent hypergeometric function [1, 6.9]. Coefficients corresponding to different ranges of the independent variable as well as those for other functions, e.g., $J_\nu(x)$ and $Y_\nu(x)$, are under construction and the present tables are selected examples only. The expansions are readily evaluated using a nesting procedure described in [4], [7]. For similar expansions, see [7], and for many Chebyshev expansions of functions over a finite interval, see [2]–[6] and the references given there. The number in parenthesis after each entry in the tables is the power of 10 by which the entry is to be multiplied.

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EXPANSION FORMULAS FOR GENERALIZED HYPERGEOMETRIC FUNCTIONS (1)

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INTRODUCTION

In a previous paper [1], (2) expansions of hypergeometric functions in series of hypergeometric functions were derived. The G-function [2], [3], a generalization of the hypergeometric function, can be defined by a Mellin-Barnes contour integral or represented as a sum of hypergeometric functions. A natural question is whether expansions of G-functions in series of G-functions exist which are generalizations of the results given in [1]. The answer is yes. Indeed, expansions of this kind given by Meijer [4] are special cases of our main theorems.

In section 1 of this paper, we generalize the results in [1] to include a multiplier of a power of the independent variable, and also present conditions for the validity of our expansions, which were not given in [1].

In section 2 of this paper, we use a representation of the G-function as a sum of hypergeometric functions to prove two theorems which generalize the expansions given by Meijer.

Section 3 contains several examples based on the theorems of section 1.

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(2) Numbers in square brackets pertain to references at end of paper.

1. Preliminary Results

The hypergeometric function ${}_pF_q(z)$, [3, Ch. 4], is defined by

$$(1.1)\quad \begin{cases} {}_pF_q(z) = {}_pF_q\left(\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}\middle| z\right) = \sum_{k=0}^{\infty} \dfrac{\prod_{j=1}^{p} (a_j)_k}{\prod_{j=1}^{q} (b_j)_k} \cdot \dfrac{z^k}{k!}, \\ \text{where} \quad (\sigma)_\mu = \dfrac{\Gamma(\sigma+\mu)}{\Gamma(\sigma)}. \end{cases}$$

We assume that no a_j is equal to any b_j and that no b_j is a negative intege For ease in writing, we employ the contracted notation

$$(1.2)\quad {}_pF_q(z) = {}_pF_q\left(\begin{matrix} a_p \\ b_q \end{matrix}\middle| z\right) = \sum_{k=0}^{\infty} \frac{(a_p)_k}{(b_q)_k} \cdot \frac{z^k}{k!}.$$

Thus $(a_p)_k$ is to be interpreted as $\Pi_{j=1}^{p}(a_j)_k$ and similarly for $(b_q)_k$. Considere as a power series in z, ${}_pF_q(z)$ has a radius of convergence equal to infinity $p \leq q$ and equal to unity if $p = q+1$. In general, ${}_pF_q(z)$ is not defined $p \geq q+2$.

We first prove a

Lemma: If $\mu \geq 0$, $0 < \omega < 1$, $\alpha > -1$, $\beta > -1$, then

$$(1.3)\quad \begin{cases} \omega^\mu e^{z\omega} = (\beta+1)_\mu \sum_{n=0}^{\infty} \dfrac{(2n+\alpha+\beta+1)(-\mu)_n}{n!\,(n+\alpha+\beta+1)_{\mu+1}}\, {}_2F_2\left(\begin{matrix} \mu+1, \mu+\beta+1 \\ \mu-n+1, \mu+n+\alpha+\beta+2 \end{matrix}\middle| z\right) \times \\ \qquad \times\, {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \beta+1 \end{matrix}\middle| \omega\right). \end{cases}$$

Remark: If μ is a positive integer, this can be put in a more attractiv form. See the discussion following (1.16).

Proof: Let

$$(1.4)\quad \begin{cases} f(x) = \left(\dfrac{1+x}{2}\right)^\mu e^{z(1+x)/2} = \sum_{n=0}^{\infty} C_n P_n^{(\alpha,\beta)}(x), \quad -1 < x < 1, \\ P_n^{(\alpha,\beta)}(x) = \dfrac{(-)^n(\beta+1)_n}{n!}\, {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \beta+1 \end{matrix}\middle| \dfrac{1+x}{2}\right). \end{cases}$$

Here $P_n^{(\alpha,\beta)}(x)$ is a Jacobi polynomial [3, Ch. 10]. Equation (1.4) is valid since $f(x$ is continuous and of bounded variation in the open interval $(-1, 1)$ when $\mu \geq 0$

Now multiply both sides of (1.4) by $(1-u)^{\alpha}(1+u)^{\beta} P_m^{(\alpha,\beta)}(u)$, $\alpha > -1$, $\beta > -1$, and integrate from -1 to 1. Using the orthogonality property of the Jacobi polynomials, we get

$$(1.5) \qquad \begin{cases} C_n = h_n^{-1} 2^{-\mu} \displaystyle\int_{-1}^{1} (1+x)^{\mu+\beta}(1-x)^{\alpha} e^{z(1+x)/2} P_n^{(\alpha,\beta)}(x)\,dx, \\ h_n = \dfrac{2^{\alpha+\beta+1}\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{(2n+\alpha+\beta+1)\,n!\,\Gamma(n+\alpha+\beta+1)}. \end{cases}$$

In (1.5) expand $e^{z(1+x)/2}$ in powers of $(1+x)$ and integrate termwise. Then using a known result [5, 16.4 (1)] we readily evaluate C_n, and (1.3) follows upon replacing x in (1.4) by $2\omega - 1$.

Theorem I.

Assumptions:

1. Let none of the following quantities be negative integers:

μ; $a_j + \mu - 1$, $j = 1, 2, \ldots, t$; $c_j - \mu - 1$, $j = 1, 2, \ldots, r$; γ; $\beta_j - 1$, $j = 1, 2, \ldots, u$.

2. Let p, q, r and s be positive integers or zero and

$$\text{(a)} \quad \begin{cases} p+r \leqslant q+s & \text{or} \quad p+r = q+s+1 \quad \text{if} \quad |z\omega| < 1, \\ p+t \leqslant q+u+1 & \text{or} \quad p+t = q+u+2 \quad \text{if} \quad |z| < 1, \end{cases}$$

$$\text{(b)} \quad r+u+1 = s+t.$$

3. Let $0 < \omega < 1$.

4. Let the following inequalities be satisfied.

$$\mu(s-r-2) + \sum_{j=1}^{r} c_j + \sum_{j=1}^{u} \beta_j - \sum_{j=1}^{t} \alpha_j - \sum_{j=1}^{s} d_j < \frac{1}{2};$$

$$c_j > 0,\ j = 1, 2, \ldots, r; \quad \beta_j + \mu > 0,\ j = 1, 2, \ldots, u.$$

Then

$$(1.6) \qquad \begin{cases} \omega^{\mu}\, {}_{p+r}F_{q+s}\left(\begin{matrix} a_p, \ c_r \\ b_q, \ d_s \end{matrix}\middle| z\omega\right) = \dfrac{(c_r)_{-\mu}(a_t)_{\mu}}{(\beta_u)_{\mu}(d_s)_{-\mu}} \displaystyle\sum_{n=0}^{\infty} \dfrac{(2n+\gamma)(-\mu)_n}{n!(n+\gamma)_{\mu+1}} \times \\ \times\, {}_{p+t+1}F_{q+u+2}\left(\begin{matrix} \mu+1, \ \alpha_t+\mu, \ a_p \\ \mu-n+1, \ n+\gamma+\mu+1, \ \beta_u+\mu, \ b_q \end{matrix}\middle| z\right) \times \\ \qquad \times\, {}_{r+u+2}F_{s+t}\left(\begin{matrix} -n, \ n+\gamma, \ c_r-\mu, \ \beta_u \\ \alpha_t, \ d_s-\mu \end{matrix}\middle| \omega\right). \end{cases}$$

Proof: We first prove (1.6) for the case where $u = 0$, $t = 1$ and $\alpha_1 = c$
That is,

$$(1.7)\quad \begin{cases} \omega^{\mu}\,{}_{p+r}F_{q+s}\left(\begin{matrix}a_p,\ c_r\\ b_q,\ d_s\end{matrix}\middle|\, z\,\omega\right) = \dfrac{(c_r)_{-\mu}(\alpha)_{\mu}}{(d_s)_{-\mu}} \displaystyle\sum_{n=0}^{\infty} \dfrac{(2n+\gamma)(-\mu)_n}{n!\,(n+\gamma)_{\mu+1}} \times \\ \times\ {}_{p+2}F_{q+2}\left(\begin{matrix}\mu+1,\ \alpha+\mu,\ a_p\\ \mu-n+1,\ n+\gamma+\mu+1,\ b_q\end{matrix}\middle|\, z\right) {}_{r+2}F_{s+1}\left(\begin{matrix}-n,\ n+\gamma,\ c_r-\mu\\ \alpha,\ d_s-\mu\end{matrix}\middle|\, \omega\right)\end{cases}$$

Our proof is by induction on the four parameters p, q, r and s. (Note that th case $p = q = r = s = 0$ is the statement (1.3) if $\gamma = \alpha + \beta + 1$). Multiply (1.7 by $\omega^{\sigma-\mu-1}$ and evaluate the Laplace transform $\int_0^{\infty} e^{-\lambda\omega}\,\omega^{\sigma-1}\,{}_{p+r}F_{q+s}\left(\begin{matrix}a_p,\ c_r\\ b_q,\ d_s\end{matrix}\middle|\, z\,\omega\right) d$ using a known result [5, 4.23 (17)]. Then

$$(1.8)\quad \begin{cases} \lambda^{-\mu}\,{}_{p+r+1}F_{q+s}\left(\begin{matrix}a_p,\ c_r,\ \sigma\\ b_q,\ d_s\end{matrix}\middle|\, \dfrac{z}{\lambda}\right) = \dfrac{(\sigma)_{-\mu}(c_r)_{-\mu}(\alpha)_{\mu}}{(d_s)_{-\mu}} \displaystyle\sum_{n=0}^{\infty} \dfrac{(2n+\alpha)(-\mu)_n}{n!\,(n+\gamma)_{\mu+1}} \times \\ \times\ {}_{p+2}F_{q+2}\left(\begin{matrix}\mu+1,\ \alpha+\mu,\ a_p\\ \mu-n+1,\ n+\gamma+\mu+1,\ b_q\end{matrix}\middle|\, z\right) {}_{r+3}F_{s+1}\left(\begin{matrix}-n,\ n+\gamma,\ c_r-\mu,\ \sigma-\mu\\ \alpha,\ d_s-\mu\end{matrix}\middle|\, \dfrac{1}{\lambda}\right)\end{cases}$$

Now replace λ by ω^{-1} and σ by c_{r+1}, and the induction on r is completed. T effect the induction with respect to s, multiply both sides of (1.7) by $\omega^{\sigma-\mu}$ replace ω by λ^{-1}, take the inverse Laplace transform [5, 5.21 (1)] of both sides and identify σ with d_{s+1}. If these Laplace transform techniques are applied to instead of ω, the inductions on p and q follow.

To show (1.6), equate the coefficients of z^k on both sides of (1.6). W may show that the relation so obtained follows from the proved result (1.7 in the following manner. In (1.7) put $z = 0$. Replace r by $r + u$, let $c_{r+m} = \beta_m + \mu$ $m = 1, 2, \ldots, u$ and replace c_m by $c_m + k$, $m = 1, 2, \ldots, r$. Replace s b $s + t$, let $d_{s+m} = \alpha_m + \mu$, $m = 1, 2, \ldots, t$ and replace d_m by $d_m + k$, $m = 1, 2, \ldots, s$ Replace ω by $\alpha\,\omega$ and let $\alpha \to \infty$. Finally, replace μ by $\mu + k$ and this com pletes the identification.

We have so far shown that (1.6) is a formal identity. We now prove tha (1.6) is defined and converges under the stated hypotheses.

The assumptions 1 are needed to insure that the gamma functions in (1.6) which appear outside the hypergeometric functions are finite. The necessary conditions to insure the convergence and meaning of the hypergeometric functions in (1.6) arise from the remarks following (1.2) and are covered by assumptions 2(a). However, in (1.6) any of the denominator parameters in the hypergeometric functions can be a negative integer or zero. To show this, suppose a particular b_q, call it b_m, is a negative integer. Multiply both sides of (1.6) by $\{\Gamma(b_m)\}^{-1}$ and it is easy to see that the resulting equation has meaning. The other denominator parameters are treated in a similar fashion. Also, μ may be a positive integer. For a statement of (1.6) when μ is a positive integer, see (1.18).

The remaining hypotheses arise from the consideration of the convergence of the infinite series (1.6), and depend on the asymptotic nature of the hypergeometric functions ${}_{p+t+1}F_{q+u+2}$ and ${}_{r+u+2}F_{s+t}$ for large n. The situation for the former function is easily disposed of. Indeed the assumptions 2(a) guarantee that the ${}_{p+t+1}F_{q+u+2}$ converges and for z fixed, we have

$$(1.9)\quad {}_{p+t+1}F_{q+u+2}\left(\begin{matrix}\mu+1,\ \alpha_t+\mu,\ a_p\\ \mu-n+1,\ n+\gamma+\mu+1,\ \beta_u+\mu,\ b_q\end{matrix}\middle| z\right)=1+0(n^{-2}).$$

We now consider the behavior of

$$(1.10)\quad \begin{cases} H_n(\omega)=A_nB_n(\omega),\\ A_n=\dfrac{(2n+\gamma)(-\mu)_n}{n!\,(n+\gamma)_{\mu+1}},\quad B_n(\omega)={}_{r+u+2}F_{s+t}\left(\begin{matrix}-n,\ n+\gamma,\ c_r-\mu,\ \beta_u\\ \alpha_t,\ d_s-\mu\end{matrix}\middle|\,\omega\right).\end{cases}$$

Since

$$(1.11)\quad \Gamma(a+n)/\Gamma(b+n)=n^{a-b}[1+0(n^{-1})],$$

$$(1.12)\quad A_n=\frac{2n^{-1-2\mu}}{\Gamma(-\mu)}[1+0(n^{-1})].$$

The analysis for $B_n(\omega)$ for large n follows from [6], [7]. We first consider the case when $r+u+1=s+t$. Using (1.11) and Eq. (2.5) of [6], we get

$$(1.13)\quad \begin{cases} B_n(\omega)\sim\displaystyle\sum_{j=1}^{r}n^{-2(c_j-\mu)}\mathcal{L}^{(c_j-\mu)}_{r+u+2,\,s+t}(\omega)+\sum_{j=1}^{u}n^{-2\beta_j}\mathcal{L}^{(\beta_j)}_{r+u+2,\,s+t}(\omega)+\\ \qquad+\dfrac{\Gamma(\alpha_t)\Gamma(d_s-\mu)\,n^{2g}}{\Gamma(c_r-\mu)\Gamma(\beta_u)\Gamma\left(\frac{1}{2}\right)}K_n(\omega),\end{cases}$$

where $|K_n(\omega)|$ is bounded for all n if $0 < \omega < 1$, and

$$(1.14) \qquad 2g = \frac{1}{2} + \sum_{j=1}^{r} c_j + \sum_{j=1}^{u} \beta_j - \sum_{j=1}^{t} \alpha_j - \sum_{j=1}^{s} d_j + \mu(s-r).$$

For the precise nature of the $\mathcal{L}^{(\delta_j)}_{r+u+2,s+t}(\omega)$ terms, see the reference cited. Fo our present purpose, we only need the order estimate

$$(1.15) \qquad \mathcal{L}^{(\delta_j)}_{r+u+2,s+t}(\omega) = \xi\omega^{-\delta_j}[1 + 0(n^{-2})],$$

where ξ is independent of n and ω. To achieve this result, we suppose tha the $\mathcal{L}^{(\delta_j)}_{r+u+2,s+t}(\omega)$ terms in (1.13) are linearly independent. Such is the case i none of the numerator parameters with the exception of n and $n+\gamma$ in $B_n(\omega)$ see (1.10), differ by an integer. If only two of the parameters do differ by ar integer m, say $\beta_2 = \beta_1 + m$, then a limit process must be used to evaluate the linear combination of the $\mathcal{L}^{(\beta_1)}_{r+u+2,s+t}(\omega)$ and $\mathcal{L}^{(\beta_2)}_{r+u+2,s+t}(\omega)$ terms. Our discussior here is necessarily brief. For a rather complete treatment of the subject see [6 and [12, 1.3.4] (1). In the latter reference, the necessary limit procedure is demonstrated for a closely related function. If the limit process is carried out, then (1.13) remains unchanged except that $\sum_{j=1}^{u}$ is replaced by $\sum_{j=3}^{u}$ plus the term $n^{-2\beta_2}(l\,n\,n)[1 + 0(n^{-2})]$. This does not alter the convergence of the series $\sum_{n=0}^{\infty} H_n(\omega)$ since we require that $H_n(\omega)$ behave like $n^{-1-\delta}[1 + 0(1/n)]$, $\delta > 0$. Thus assumptions 4 are sufficient for convergence (2).

If assumption 2 (b) is not satisfied, an analysis similar to the above, but based on results given in [7], shows that the series $\sum_{n=0}^{\infty} H_n(\omega)$ in general diverges. This completes the proof of Theorem I.

If $\mu = m$ is a positive integer or zero, (1.6) can be put in a more attractive form.

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(1) In Eq. 1.3.4 (2) of this reference for $-\psi(1+\beta_q+k)+\psi(1+\beta_q)$ read $-\psi(1+\delta_s+k)+$ $+\psi(1+\delta_s)$. Also in 1.3.4 (7), for the denominator parameter $b_q + m$ of the ${}_{p+1}F_{q+1}$, read $b_q + 1 + m$.

(2) If more than two of the numerator parameters in $B_n(\omega)$ differ by an integer, see the concluding remarks of 1.3.4 of [12]. Such singular situations do not invalidate our analysis.

Corollary. Under the hypotheses of Theorem I,

$$
(1.16)\quad \begin{cases}
\omega^m {}_{p+r}F_{q+s}\left(\begin{matrix} a_p,\ c_r \\ b_q,\ d_s \end{matrix}\middle| z\omega\right) = \dfrac{(c_r)_{-m}(\alpha_t)_m}{(\beta_u)_m (d_s)_{-m}} \displaystyle\sum_{n=0}^{m-1} \dfrac{(2n+\gamma)(-m)_n}{n!\,(n+\gamma)_{m+1}} \\
\times {}_{p+t+1}F_{q+u+2}\left(\begin{matrix} m+1,\ \alpha_t+m,\ a_p \\ m-n+1,\ m+n+\gamma+1,\ \beta_u+m,\ b_q \end{matrix}\middle| z\right) {}_{r+u+2}F_{s+t}\left(\begin{matrix} -n,\ n+\gamma,\ c_r-m,\ \beta_u \\ \alpha_t,\ d_s-m \end{matrix}\middle| \omega\right) \\
+ \dfrac{(-)^m (c_r)_{-m}(\alpha_t)_m}{(\beta_u)_m (d_s)_{-m}} \displaystyle\sum_{n=0}^{\infty} \dfrac{(-)^n (\alpha_t+m)_n (a_p)_n z^n}{n!\,(\beta_u+m)_n (b_q)_n (n+m+\gamma)_{m+n}} \\
\times {}_{p+t+1}F_{q+u+2}\left(\begin{matrix} m+n+1,\ \alpha_t+m+n,\ a_p+n \\ 2n+\gamma+2m+1,\ \beta_u+m+n,\ n+1,\ b_q+n \end{matrix}\middle| z\right) \\
\times {}_{r+u+2}F_{s+t}\left(\begin{matrix} -m-n,\ m+n+\gamma,\ c_r-m,\ \beta_u \\ \alpha_t,\ d_s-m \end{matrix}\middle| \omega\right).
\end{cases}
$$

Note that if $m=0$, the first summation is nil. Equation (1.16) generalizes a previous statement given in [1]. Also if $p=q=r=s=0$, we get (1.3) when $\mu=m$.

We now derive two further results similar to Theorem I. These are confluent forms of (1.6) and (1.16) which follow upon using the fact that

$$
(1.17)\qquad \lim_{a\to\infty} (a)_n (z/a)^n = z^n .
$$

Thus in (1.6), replace ω by ω/γ, replace z by γz and let $\gamma \to \infty$. Then we have

Theorem II.

Assumptions:

1. Let none of the following quantities be negative integers:

μ; $\alpha_j+\mu-1$, $j=1, 2, \ldots, t$; $c_j-\mu-1$, $j=1, 2, \ldots, r$; β_j-1, $j=1, 2, \ldots, u$.

2. Let p, q, r and s be positive integers or zero and

(a) $p+r\leqslant q+s$ or $p+r=q+s+1$ if $|z\omega|<1$, $p+t\leqslant q+u$,

(b) $r+u+1=s+t$.

3. Let $0<\omega<\infty$.

4. Let the following inequalities be satisfied.

$$
\mu(s-r-2)+\sum_{j=1}^{r} c_j+\sum_{j=1}^{u}\beta_j-\sum_{j=1}^{t}\alpha_j-\sum_{j=1}^{s} d_j < -\frac{1}{2};
$$

$c_j>0$, $j=1, 2, \ldots, r$; $\beta_j+\mu>0$, $j=1, 2, \ldots, u$.

Then

$$(1.18)\quad \left\{\begin{aligned} &\omega^{\mu}\,{}_{p+r}F_{q+s}\left(\begin{matrix}a_p,\ c_r\\ b_q,\ d_s\end{matrix}\middle| z\omega\right) = \frac{(c_r)_{-\mu}(\alpha_t)_{\mu}}{(\beta_u)_{\mu}(d_s)_{-\mu}}\sum_{n=0}^{\infty}\frac{(-\mu)_n}{n!}\\ &\times {}_{p+t+1}F_{q+u+1}\left(\begin{matrix}\mu+1,\ \alpha_t+\mu,\ a_p\\ \mu-n+1,\ \beta_u+\mu,\ b_q\end{matrix}\middle| z\right){}_{r+u+1}F_{s+t}\left(\begin{matrix}-n,\ c_r-\mu,\ \beta_u\\ \alpha_t,\ d_s-\mu\end{matrix}\middle|\omega\right)\end{aligned}\right.$$

Proof: The order estimate

$$_{p+t+1}F_{q+u+1}\left(\begin{matrix}\mu+1,\ \alpha_t+\mu,\ a_p\\ \mu-n+1,\ \beta_u+\mu,\ b_q\end{matrix}\middle| z\right) = 1 + 0(1/n)$$

can be readily deduced from [8]. The behavior of the $_{r+u+1}F_{s+t}$ in (1.18) follows from [7]. From this point on, the proof is very similar to that of Theorem I, and we omit details. Under the same assumptions as Theorem II, we also have the following

Corollary.

$$(1.19)\quad \left\{\begin{aligned} &\omega^{m}\,{}_{p+r}F_{q+s}\left(\begin{matrix}a_p,\ c_r\\ b_q,\ d_s\end{matrix}\middle| z\omega\right) = \frac{(c_r)_{-m}(\alpha_t)_{m}}{(\beta_u)_{m}(d_s)_{-m}}\sum_{n=0}^{m-1}\frac{(-m)_n}{n!}\\ &\times {}_{p+t+1}F_{q+u+1}\left(\begin{matrix}m+1,\ \alpha_t+m,\ a_p\\ m-n+1,\ \beta_u+m,\ b_q\end{matrix}\middle| z\right){}_{r+u+1}F_{s+t}\left(\begin{matrix}-n,\ c_r-m,\ \beta_u\\ \alpha_t,\ d_s-m\end{matrix}\middle|\omega\right)\\ &\qquad + \frac{(-)^m(c_r)_{-m}(\alpha_t)_m}{(\beta_u)_m(d_s)_m}\sum_{n=0}^{\infty}\frac{(-)^n(\alpha_t+m)_n(a_p)_n z^n}{n!(\beta_u+m)_n(b_q)_n}\\ &\times {}_{p+t+1}F_{q+u+1}\left(\begin{matrix}m+n+1,\alpha_t+m+n,a_p+n\\ n+1,\beta_u+m+n,b_q+n\end{matrix}\middle| z\right){}_{r+u+1}F_{s+t}\left(\begin{matrix}-n-m,c_r-m,\beta_u\\ \alpha_t,d_s-m\end{matrix}\middle|\omega\right).\end{aligned}\right.$$

2. Expansions of G-Functions

To arrive at our main theorem, we have need for the representation of the G-function as a series of hypergeometric functions, see [3, 5.3 (5)]. For our present purposes, it is convenient to employ the representation in the form

$$(2.1)\quad \left\{\begin{aligned} &G^{m,k+r}_{p+r,q+s}\left(z\omega\middle|\begin{matrix}c_s,\ a_p\\ b_q,\ d_s\end{matrix}\right) = \sum_{h=1}^{m}\frac{\prod_{j=1}^{m}{}'\,\Gamma(b_j-b_h)\prod_{j=1}^{r}\Gamma(1+b_h-c_j)\prod_{j=1}^{k}\Gamma(1+b_h-a_j)}{\prod_{j=m+1}^{q}\Gamma(1+b_h-b_j)\prod_{j=k+1}^{p}\Gamma(a_j-b_h)\prod_{j=1}^{s}\Gamma(1+b_h-d_j)}\\ &\times (z\omega)^{b_h}\,{}_{p+r}F_{q+s-1}\left(\begin{matrix}1+b_h-c_r,\ 1+b_h-a_p\\ \{1+b_h-b_q\}',\ 1+b_h-d_s\end{matrix}\middle|(-)^{k+m+p}z\omega\right)\end{aligned}\right.$$

where $m \leqslant q$ and the parameters are such that the gamma functions in the numerator are finite. Also, the prime in $\prod_{j=1}^{m}{}'$ and $\{1 + b_h - b_q\}'$ means to omit the terms when $h = j$. We now prove

Theorem III.

Assumptions:

1. Let none of the following quantities be negative integers.

b_h; $\alpha_j + b_h - 1$, $j = 1, 2, \ldots, t$; $-c_j$, $j = 1, 2, \ldots, r$; γ; $\beta_j - 1$, $j = 1, 2, \ldots, u$; $b_h - c_j$, $j = 1, 2, \ldots, r$; $b_h - a_j$, $j = 1, 2, \ldots, k$. Here and in what follows $h = 1, 2, \ldots, m$ always.

2. Let p, q, r, s, t, u, k and m be positive integers or zero.

(a) $$\begin{cases} p + r \leqslant q + s - 1 \quad \text{or} \quad p + r = q + s & \text{if } |z\omega| < 1, \\ p + t \leqslant q + u \quad \text{or} \quad p + t = q + u + 1 & \text{if } |z| < 1, \end{cases}$$

(b) $r + u + 1 = s + t$,

(c) $0 \leqslant m \leqslant q$; $0 \leqslant k \leqslant p$; $q + s \geqslant 1$.

3. Let $0 < \omega < 1$, $z \neq 0$.

4. $$\sum_{j=1}^{s} d_j - \sum_{j=1}^{r} c_j + \sum_{j=1}^{u} \beta_j - \sum_{j=1}^{t} \alpha_j - 2b_h < s - r + \frac{1}{2},$$

$$1 + b_h - c_j > 0, \; j = 1, 2, \ldots, r; \;\; \beta_j + b_h > 0, \; j = 1, 2, \ldots, u.$$

Then

$$\left\{ \begin{aligned} & G_{p+r,q+s}^{m,k+r}\left(z\omega \middle| \begin{matrix} c_r, a_p \\ b_q, d_s \end{matrix}\right) = \frac{\Gamma(1-c_r)\,\Gamma(\beta_u)}{\Gamma(\alpha_t)\,\Gamma(1-d_s)} \sum_{n=0}^{\infty} \frac{(-)^n (2n+\gamma)\,\Gamma(n+\gamma)}{n!} \\ & \times G_{p+t+1,q+u+2}^{m,k+t+1}\left(z \middle| \begin{matrix} 0, 1-\alpha_t, a_p \\ b_q, 1-\beta_u, -n-\gamma, n \end{matrix}\right) \times {}_{r+u+2}F_{s+t}\left(\begin{matrix} -n, n+\gamma, 1-c_r, \beta_u \\ \alpha_t, 1-d_s \end{matrix} \middle| \omega\right). \end{aligned} \right. \tag{2.2}$$

Proof:

The proof proceeds by applying Theorem I to each of the hypergeometric functions on the right-hand side of (2.1). Thus in (1.6) replace a_p by $1 + b_h - a_p$, c_r by $1 + b_h - c_r$, b_q by $1 + b_h - b_q$, q, by $q - 1$, d_s by $1 + b_h - d_s$, z by $(-)^{k+m+p} z$ and μ by b_h. Employ this formula in (2.1), and the resulting series may be regrouped and expressed as a series of G-functions which leads to the assertion (2.2).

All the above assumptions follow from those in Theorem I except 1 and 2 (c) which are necessary to insure that the expansion (2.2) has meaning.

Theorem IV.

Assumptions:

1. Let none of the following quantities be negative integers. b_h; $\alpha_j + b_h - 1$, $j = 1, 2, \ldots, t$; $-c_j$, $j = 1, 2, \ldots, r$; $\beta_j - 1$, $j = 1, 2, \ldots, u$; $b_h - c_j$, $j = 1, 2, \ldots, r$; $b_h - a_j$, $j = 1, 2, \ldots, k$. Here and in what follows $h = 1, 2, \ldots, m$ always.

2. Let p, q, r, s, t, u, k and m be positive integers or zero.

(a) $p + r \leqslant q + s - 1$ or $p + r = q + s$ if $|z\omega| < 1$, $p + t \leqslant q + u - 1$

(b) $r + u + 1 = s + t$,

(c) $0 \leqslant m \leqslant q$; $0 \leqslant k \leqslant p$; $q + s \geqslant 1$.

3. Let $0 < \omega < \infty$, $z \neq 0$.

4. $\sum_{j=1}^{s} d_j - \sum_{j=1}^{r} c_j + \sum_{j=1}^{u} \beta_j - \sum_{j=1}^{t} \alpha_j - 2b_h < s - r - \frac{1}{2}$,

$1 + b_h - c_j > 0$, $j = 1, 2, \ldots, r$; $\beta_j + b_h > 0$, $j = 1, 2, \ldots, u$.

Then

$$
(2.3) \quad \left\{
\begin{aligned}
&G^{m,k+r}_{p+r,q+s}\left(z\omega\middle|\begin{matrix}c_r,\ a_p\\ b_q,\ d_s\end{matrix}\right) = \frac{\Gamma(1-c_r)\Gamma(\beta_u)}{\Gamma(1-d_s)\Gamma(\alpha_t)}\sum_{n=0}^{\infty}\frac{(-)^n}{n!}\\
&\times G^{m,k+t+1}_{p+t+1,q+u+1}\left(z\middle|\begin{matrix}0,\ 1-\alpha_t,\ a_p\\ b_q,\ 1-\beta_u,\ n\end{matrix}\right) \times {}_{r+u+1}F_{s+t}\left(\begin{matrix}-n,\ 1-c_r,\ \beta_u\\ \alpha_t,\ 1-d_s\end{matrix}\middle|\omega\right).
\end{aligned}
\right.
$$

Proof:

The proof is very similar to that of Theorem III, and follows by applying Theorem II (with appropriate changes in notation) to the hypergeometric functions on the right-hand side of (2.1). A formal proof follows by using the confluence principle in Theorem III. That is, replace z by γz, ω by ω/γ and let $\gamma \to \infty$.

We now state two further theorems which follow directly from Theorems III and IV by applying first the identity [3, 5.3.1 (9)]

$$
(2.4) \quad G^{m,n}_{p,q}\left(x\middle|\begin{matrix}a_p\\ b_q\end{matrix}\right) = G^{n,m}_{q,p}\left(x^{-1}\middle|\begin{matrix}1-b_q\\ 1-a_p\end{matrix}\right),
$$

and then

$$G^{m+1,k}_{p+1,q+1}\left(x\,\middle|\,\begin{matrix}a_p,\ 1-n\\ 1,\ b_q\end{matrix}\right)=(-)^n\,G^{m,k+1}_{p+1,q+1}\left(x\,\middle|\,\begin{matrix}1-n,\ a_p\\ b_q,\ 1\end{matrix}\right), \tag{2.5}$$

which is apparent from the definition of the G-function.

Theorem V.

Assumptions:

1. Let none of the following quantities be negative integers.

$b_j-a_h,\ j=1,2,\dots,m;\ \alpha_j-a_h,\ j=1,2,\dots,t;\ \alpha_j-1,\ j=1,2,\dots,t;$
$\gamma;\ \beta_j-1,\ j=1,2,\dots,u;\ d_j-1,\ j=1,2,\dots,s;\ d_j-a_h,\ j=1,2,\dots,s;$
$b_j-1,\ j=1,2,\dots,m$. Here and in what follows $h=1,2,\dots,k$.

2. p, q, r, s, t, u, k and m are positive integers or zero.

(a) $\begin{cases} q+s\leqslant p+r-1 & \text{or} \quad q+s=p+r \quad \text{if} \quad |z\omega|>1,\\ q+t\leqslant p+u & \text{or} \quad q+t=p+u+1 \quad \text{if} \quad |z|>1,\end{cases}$

(b) $s+u+1=r+t$,

(c) $0\leqslant k\leqslant p;\ 0\leqslant m\leqslant q;\ p+r\geqslant 1$.

3. $1<\omega<\infty$.

4. $\sum_{j=1}^{s}d_j-\sum_{j=1}^{r}c_j+\sum_{j=1}^{u}\beta_j-\sum_{j=1}^{t}\alpha_j+2a_h<5/2$,

$1-a_h+d_j>0,\ j=1,2,\dots,s;\quad 1-a_h+\beta_j>0,\ j=1,2,\dots,u.$

Then

$$\left\{\begin{aligned}&G^{m+s,k}_{p+r,q+s}\left(z\omega\,\middle|\,\begin{matrix}a_p,\ c_r\\ d_s,\ b_q\end{matrix}\right)=\frac{\Gamma(d_s)\,\Gamma(\beta_u)}{\Gamma(\alpha_t)\,\Gamma(c_r)}\sum_{n=0}^{\infty}\frac{(2n+\gamma)\,\Gamma(n+\gamma)}{n!}\\ &\times G^{m+t,k+1}_{p+u+2,q+t+1}\left(z\,\middle|\,\begin{matrix}1-n,\ a_p,\ \beta_u,\ \gamma+1+n\\ \alpha_t,\ b_q,\ 1\end{matrix}\right){}_{s+u+2}F_{r+t}\left(\begin{matrix}-n,\ n+\gamma,\ d_s,\ \beta_u\\ \alpha_t,\ c_r\end{matrix}\,\middle|\,\omega^{-1}\right).\end{aligned}\right. \tag{2.6}$$

Proof: Apply (2.4) to both sides of (2.2). Replace $1-b_q$, $1-a_p$, $1-d_s$ and $1-c_r$ by a_p, b_q, c_r and d_s, respectively. Interchange p and q, r and s, and m and k. Finally, replace z by z^{-1}, ω by ω^{-1}, and apply (2.5) to the G-function on the right-hand side. We also have the analogue of Theorem IV,

Theorem VI.

Assumptions:

1. Let none of the following quantities be negative integers.

$b_j-a_h,\ j=1,2,\dots,m;\ \alpha_j-a_h,\ j=1,2,\dots,t;\ \alpha_j-1,\ j=1,2,\dots,t;$
$\beta_j-1,\ j=1,2,\dots,u;\ d_j-1,\ j=1,2,\dots,s;\ d_j-a_h,\ j=1,2,\dots,s;$
$b_j-1,\ j=1,2,\dots,m$. Here and in what follows $h=1,2,\dots,k$.

2. p, q, r, s, t, u, k and m are positive integers or zero.

(a) $\begin{cases} q+s \leqslant p+r-1 \quad \text{or} \quad q+s=p+r \quad \text{if} \quad |z\omega|>1, \\ q+t \leqslant p+u-1, \end{cases}$

(b) $s+u+1=r+t$,

(c) $0 \leqslant m \leqslant q;\ 0 \leqslant k \leqslant p;\ p+r \geqslant 1$.

3. Let $0<\omega<\infty$.

4. $\sum_{j=1}^{s} d_j - \sum_{j=1}^{r} c_j + \sum_{j=1}^{u} \beta_j - \sum_{j=1}^{t} \alpha_j + 2a_h < 3/2$,

$1-a_h+d_j>0,\ j=1,2,\ldots,s;\ 1-a_h+\beta_j>0,\ j=1,2,\ldots,u.$

Then

$$\tag{2.7} \begin{cases} G_{p+r,q+s}^{m+s,k}\left(z\omega\middle|\begin{matrix} a_p,\ c_r \\ d_s,\ b_q\end{matrix}\right) = \dfrac{\Gamma(d_s)\Gamma(\beta_u)}{\Gamma(\alpha_t)\Gamma(c_r)}\displaystyle\sum_{n=0}^{\infty}\frac{1}{n!} \\ \times G_{p+u+1,q+t+1}^{m+t,k+1}\left(z\middle|\begin{matrix}1-n,\ a_p,\ \beta_u \\ \alpha_t,\ b_q,\ 1\end{matrix}\right)\,{}_{s+u+1}F_{r+t}\left(\begin{matrix}-n,\ d_s,\ \beta_u \\ \alpha_t,\ c_r\end{matrix}\middle|\omega^{-1}\right). \end{cases}$$

Proof: The proof follows exactly the proof of Theorem V, except the starting point is Theorem IV; the same identification of parameters is made, and we omit details.

The Theorems III-VI generalize expansions given previously by Meijer. For instance, his most inclusive result [2, v. 57, p. 83, Theorem 6 A] obtains from our Theorem IV when $t=u=0$ and $s=r+1$.

Note that the conditions for the validity of Theorems I-VI are sufficient only, and in some cases of interest, the conditions may be relaxed. Even when the expansions diverge a meaning in the asymptotic sense can often be assigned to the series which make the theorems useful both theoretically and computationally.

If we denote any of the G-functions occurring in Theorems III-VI by $G_{p,q}^{m,n}\left(x\middle|\begin{matrix}a_p\\ b_q\end{matrix}\right)$, ordinarily we would require that none of quantities (b_j-b_h-1), $j, h=1, 2, \ldots, m,\ j \neq h$, be a negative integer and likewise in Theorems V and VI for the quantities $(a_h-a_j-1),\ j,\ h=1, 2, \ldots, n,\ j\neq h$. We have deleted such hypotheses. For if one of these quantities is a negative integer, a limiting process may be invoked to obtain a meaningful expansion. We defer further remarks for a future paper.

3. EXAMPLES

We first show that (1.6) can be specialized to give expansions of hypergeometric functions in series of the shifted Jacobi polynomials

$$R_n^{(\alpha,\beta)}(\omega) = P_n^{(\alpha,\beta)}(2\omega - 1) = \frac{(-)^n(\beta+1)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n,\ n+\alpha+\beta+1\\ \beta+1\end{matrix}\middle|\omega\right). \tag{3.1}$$

Thus in (1.6) let $r = u = s = 0$, $t = 1$, $\gamma = \alpha + \beta + 1$, $\alpha_1 = \beta + 1$. Then

$$\left\{\begin{aligned} &\omega^{\mu}\,{}_pF_q\left(\begin{matrix}a_p\\ b_q\end{matrix}\middle| z\omega\right) = (\beta+1)_{\mu}\sum_{n=0}^{\infty}\frac{(-)^n(2n+\alpha+\beta+1)(-\mu)_n}{(n+\alpha+\beta+1)_{\mu+1}(\beta+1)_n}\\ &\times {}_{p+2}F_{q+2}\left(\begin{matrix}\mu+1,\ \beta+1+\mu,\ a_p\\ \mu-n+1,\ n+\alpha+\beta+\mu+2,\ b_q\end{matrix}\middle| z\right)R_n^{(\alpha,\beta)}(\omega),\end{aligned}\right. \tag{3.2}$$

provided μ, $\beta+\mu$ and $\alpha+\beta+1$ are not negative integers. Also $2\mu+\beta > -3/2$, $p \leqslant q$ or $p = q+1$ if $|z| < 1$ and $0 < \omega \leqslant 1$. Similarly from (1.5), we have

$$\left\{\begin{aligned} &\omega^{m}\,{}_pF_q\left(\begin{matrix}a_p\\ b_q\end{matrix}\middle| z\omega\right) = m(\beta+1)_m\sum_{n=0}^{m-1}\frac{(2n+\alpha+\beta+1)}{(\beta+1)_n(n+\alpha+\beta+1)_{m+1}(m-n)!}\\ &\times {}_{p+2}F_{q+2}\left(\begin{matrix}m+1,\ m+\beta+1,\ a_p\\ m-n+1,\ m+n+\alpha+\beta+2,\ b_q\end{matrix}\middle| z\right)R_n^{(\alpha,\beta)}(\omega) + \sum_{n=0}^{\infty}\frac{(m+n)!\,(a_p)_n z^n}{n!\,(m+n+\alpha+\beta+1)_{m+n}(b_q)_n}\\ &\times {}_{p+2}F_{q+2}\left(\begin{matrix}m+n+1,\ \beta+m+n+1,\ a_p+n\\ 2m+2n+\alpha+\beta+2,\ n+1,\ b_q+n\end{matrix}\middle| z\right)R_{m+n}^{(\alpha,\beta)}(\omega),\end{aligned}\right. \tag{3.3}$$

provided $\beta+m$ and $\alpha+\beta+1$ are not negative integers, $2m+\beta > -3/2$, $p \leqslant q$ or $p = q+1$ if $|z| < 1$ and $0 \leqslant \omega \leqslant 1$.

Note that in (3.2) and (3.3), if $\alpha = \beta = 0$, we get expansions in series of Legendre polynomials while if $\alpha = \beta = -1/2$, we get expansions in series of Chebyshev polynomials of the first kind. For numerous expansions of the latter type, see [9], [10], [11], [13].

We next derive from (1.18) and (1.19) expansions in series of the Laguerre polynomials

$$L_n^{(\alpha)}(\omega) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{matrix}-n\\ \alpha+1\end{matrix}\middle|\omega\right), \tag{3.4}$$

and the Hermite polynomials

$$H_{2n}(\omega) = (-)^n 2^{2n} n!\,L_n^{(-1/2)}(\omega^2), \qquad H_{2n+1}(\omega) = (-)^n 2^{2n+1} n!\,\omega\,L_n^{(1/2)}(\omega^2). \tag{3.5}$$

Thus for the Laguerre polynomials, we get

$$(3.6)\quad \omega^{\mu}{}_pF_q\left(\begin{matrix}a_p\\b_q\end{matrix}\middle| z\omega\right) = (\alpha+1)_\mu \sum_{n=0}^{\infty} \frac{(-\mu)_n}{(\alpha+1)_n}\, {}_{p+2}F_{q+1}\left(\begin{matrix}\mu+1,\ \alpha+\mu+1,\ a_p\\ \mu-n+1,\ b_q\end{matrix}\middle| z\right) L_n^{(\alpha)}(\omega)$$

provided μ and $\alpha+\mu$ are not negative integers, $\alpha+2\mu > -1/2$, $p+1 \leqslant q$ and $0 < \omega < \infty$. Also

$$(3.7)\quad \begin{cases} \omega^m{}_pF_q\left(\begin{matrix}a_p\\b_q\end{matrix}\middle| z\omega\right) = m!(\alpha+1)_m \displaystyle\sum_{n=0}^{m-1} \frac{(-)^n}{(m-n)!\,(\alpha+1)_n}\, {}_{p+2}F_{q+1}\left(\begin{matrix}m+1,\ \alpha+m+1,\ a_p\\ m-n+1,\ b_q\end{matrix}\middle| z\right) L_n^{(\alpha)}(\omega) \\ + (-)^m \displaystyle\sum_{n=0}^{\infty} \frac{(-)^n (m+n)!\,(a_p)_n z^n}{n!\,(b_q)_n}\, {}_{p+2}F_{q+1}\left(\begin{matrix}m+n+1,\ m+n+\alpha+1,\ a_p+n\\ n+1,\ b_q+n\end{matrix}\middle| z\right) L_{m+n}^{(\alpha)}(\omega) \end{cases}$$

provided $\alpha+m$ is not a negative integer, $\alpha+2m > -1/2$, $p+1 \leqslant q$ and $0 < \omega < \infty$. For expansions in Hermite polynomials, we give only the formula

$$(3.8)\quad {}_pF_q\left(\begin{matrix}a_p\\b_q\end{matrix}\middle| z\omega\right) = \sum_{n=0}^{\infty} \frac{(a_p)_n (z/2)^n}{(b_q)_n\, n!}\, {}_{2p}F_{2q}\left(\begin{matrix}\dfrac{a_p+n}{2},\ \dfrac{a_p+n+1}{2}\\[2mm] \dfrac{b_q+n}{2},\ \dfrac{b_q+n+1}{2}\end{matrix}\middle| \frac{z^2}{4^{q+1-p}}\right) H_n(\omega)$$

which is valid for $p+1 \leqslant q$ and $-\infty < \omega < \infty$. To get this split ${}_pF_q\left(\begin{matrix}a_p\\b_q\end{matrix}\middle| z\omega\right)$ into its even and odd parts. To each part which is a hypergeometric function apply (3.7) with $m=0$ and (3.5) as appropriate. The two parts are then recombined to obtain (3.8).

We now present an example of (1.16). Let $m=0$ and $r=s=0$. Let $p=1$, $q=2$ and $a_1=1/2$, $b_1=1-\nu$, $b_2=1+\nu$. Let $u=1$, $t=2$ and $\beta_1=1$, $\alpha_1=1-\nu$, $\alpha_2=1+\nu$. Put $\gamma=0$, $\omega=1$ and replace z by $-z^2$. Now [3, 7.2.7 (49)]

$$(3.9)\quad J_\mu(z)J_\nu(z) = \frac{(z/2)^{\mu+\nu}}{\Gamma(\mu+1)\Gamma(\nu+1)}\, {}_2F_3\left(\begin{matrix}\dfrac{\mu+\nu+1}{2},\ \dfrac{\mu+\nu+2}{2}\\[2mm] \mu+1,\ \nu+1,\ \mu+\nu+1\end{matrix}\middle| -z^2\right),$$

where $J_\nu(z)$ is the Bessel function of the first kind. Also [3, 4.4 (3)]

$$(3.10)\quad {}_3F_2\left(\begin{matrix}-n,\ n,\ 1\\ 1-\nu,\ 1+\nu\end{matrix}\middle| 1\right) = \frac{\nu^2}{\nu^2-n^2}.$$

Thus

$$(3.11)\quad J_{-\nu}(z)J_\nu(z) = \frac{\nu\sin\nu\pi}{\pi}\sum_{n=0}^{\infty}\frac{\varepsilon_n J_n^2(z)}{\nu^2-n^2},\quad \varepsilon_0=1;\ \varepsilon_n=2,\ n>0,$$

nd if $\nu = 1/2$, we get

3.12) $$\frac{\sin 2z}{2z} = J_0^2(z) - 2\sum_{n=1}^{\infty} \frac{J_n^2(z)}{4n^2-1}.$$

Ve now show how (3.11) and (3.12) can be used to derive expansions for

3.13) $$Si(z) = \int_0^z t^{-1} \sin t\, dt, \qquad Vi(z) = \int_0^z t^{-1}(1-\cos t)\, dt.$$

)ifferentiate (3.11) with respect to ν and then set $\nu = 1/2$. Employing a known esult for $(\partial J_\nu(z)/\partial \nu)$ for $\nu = \pm 1/2$ [12, 7.9 (18-19)], we get with the aid of (3.12),

3.14) $$Si(z) = -\sin z + 2zJ_0^2(z/2) + 4z\sum_{n=1}^{\infty} \frac{J_n^2(z/2)}{(4n^2-1)^2}.$$

`o get an expansion for $Vi(z)$, we start with the identity,

3.15) $$Vi(z) = 1 - z^{-1}\sin z + \int_0^z t^{-1}(1 - t^{-1}\sin t)\, dt.$$

'ut (3.12) in (3.15), and use the formulas [12, 1.4.2 (4), 11.2 (16)].

3.16) $$1 - J_0^2(t) = 2\sum_{k=1}^{\infty} J_k^2(t),$$

3.17) $$2n\int_0^z t^{-1} J_n^2(t)\, dt = J_n^2(z) + 2\sum_{k=1}^{\infty} J_{n+k}^2(z), \quad n > 0,$$

o obtain

3.18) $$Vi(z) = 2\sum_{n=1}^{\infty} \left[\psi\left(n+\frac{1}{2}\right) - \psi\left(\frac{1}{2}\right)\right] J_n^2(z/2).$$

For other expansions of the sine and cosine integrals and related functions n series of Bessel functions, see [12].

Numerous special cases of the general theorems developed in this paper ıre scattered throughout the literature. An adequate list of references would be ıuite lengthy. However, the references given here together with the sources ndicated in these references provide a good coverage of this subject.

Kansas City (U. S. A.), March 1963.

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Basic series corresponding to a class of hypergeometric polynomials†

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1. *Introduction.* Let $g_n(z)$ be a polynomial of degree n in z. Then there exist constants $\Pi_{k,n}$ such that

$$z^k = \sum_{n=0}^{\infty} \Pi_{k,n} g_n(z). \tag{1·1}$$

If $f(z)$ is analytic in a neighbourhood of the origin,

$$f(\lambda z) = \sum_{k=0}^{\infty} \xi_k \lambda^k z^k, \quad \xi_k = f^{(k)}(0)/k!, \tag{1·2}$$

and formal substitution of (1·1) into (1·2) yields‡

$$f(\lambda z) = \sum_{n=0}^{\infty} C_n g_n(z), \tag{1·3}$$

$$C_n = \sum_{k=0}^{\infty} \Pi_{k,n} \xi_k \lambda^k. \tag{1·4}$$

Equation (1·3) is called the basic series for $f(\lambda z)$ corresponding to the set $\{g_n(z)\}$. For a general treatment of basic series see (1).

In this paper we study basic series for the case where $g_n(z)$ is the hypergeometric polynomial ((2), § 4·1)

$$\left.\begin{aligned} G_n(z,\gamma) &= {}_{p+2}F_q\left(\begin{matrix} -n, n+\gamma, a_1, \ldots, a_p \\ b_1, \ldots, b_q \end{matrix}\middle| z\right), \\ &= \sum_{k=0}^{n} \frac{(-n)_k (n+\gamma)_k \prod_{i=1}^{p} (a_i)_k}{\prod_{i=1}^{q} (b_i)_k} \frac{z^k}{k!}, \end{aligned}\right\} \tag{1·5}$$

where
$$(\sigma)_\mu = \frac{\Gamma(\sigma+\mu)}{\Gamma(\sigma)}. \tag{1·6}$$

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‡ Note that if $\Pi_{k,n} = O(1)$ uniformly in n as $k \to \infty$, convergence of C_n is assured if $|\lambda| <$ radius of convergence of $f(z)$.

We will use the contracted notation

$$G_n(z,\gamma) = {}_{p+2}F_q\left(\begin{matrix} -n, n+\gamma, a_p \\ b_q \end{matrix}\middle| z\right) = \sum_{k=0}^{n} \frac{(-n)_k\,(n+\gamma)_k\,(a_p)_k}{(b_q)_k\,k!} z^k. \tag{1·7}$$

Thus $(a_p)_k$ is to be interpreted as $\prod_{i=1}^{p} (a_i)_k$ and similarly for $(b_q)_k$. To insure that $G_n(z,\gamma)$ is well defined and is a polynomial of order n, we assume that none of the quantities $\gamma+1$, a_i and b_j is a negative integer or zero. We also assume no a_i is equal to any b_j.

By the following limit procedure (called a confluence with respect to γ) we obtain

$$\begin{aligned} \lim_{\gamma\to\infty} G_n\left(\frac{z}{\gamma},\gamma\right) &= \lim_{\gamma\to\infty} {}_{p+2}F_q\left(\begin{matrix} -n, n+\gamma, a_p \\ b_q \end{matrix}\middle| \frac{z}{\gamma}\right) \\ &= {}_{p+1}F_q\left(\begin{matrix} -n, a_p \\ b_q \end{matrix}\middle| z\right) = G_n(z). \end{aligned} \tag{1·8}$$

Hence, $\{G_n(z,\gamma)\}$ and its confluent form $\{G_n(z)\}$ contain the orthogonal polynomials of Jacobi, Laguerre, Hermite and Bessel.

Henceforth we use the notation

$$\left.\begin{aligned} z^k &= \sum_{n=0}^{\infty} \pi_{k,n}(\gamma)\, G_n(z,\gamma), \\ f(\lambda z) &= \sum_{n=0}^{\infty} C_n(\lambda,\gamma)\, G_n(z,\gamma), \\ C_n(\lambda,\gamma) &= \sum_{k=0}^{\infty} \pi_{k,n}(\gamma)\, \xi_k \lambda^k, \end{aligned}\right\} \tag{1·9}$$

and

$$\left.\begin{aligned} z^k &= \sum_{n=0}^{\infty} \pi_{k\,n} G_n(z), \\ f(\lambda z) &= \sum_{n=0}^{\infty} C_n(\lambda)\, G_n(z), \\ C_n(\lambda) &= \sum_{k=0}^{\infty} \pi_{k\,n} \xi_k \lambda^k, \end{aligned}\right\} \tag{1·10}$$

where $\xi_k = f^{(k)}(0)/k!$.

In § 2 our main result determines (1·9) and (1·10), in § 3 we examine the case where $\{G_n(z,\gamma)\}$ are the Jacobi polynomials, and in § 4 we apply our results to obtain economic polynomial representations of several higher transcendental functions.

2. *The determination of $\pi_{k,n}(\gamma)$ and $\pi_{k,n}$.*

THEOREM. *Let none of the quantities $\gamma+1$, a_i and b_j be a negative integer or zero. Then*

$$\pi_{k,n}(\gamma) = \frac{(b_q)_k\,(-k)_n\,(2n+\gamma)}{(a_p)_k\,(n+\gamma+1)_k\,(n+\gamma)\,n!}, \tag{2·1}$$

and hence

$$C_n(\lambda,\gamma) = \frac{(b_q)_n\,(-\lambda)^n}{(a_p)_n\,(n+\gamma)_n} \sum_{k=0}^{\infty} \frac{(n+b_q)_k\,(n+1)_k\,\xi_{k+n}\lambda^k}{(n+a_p)_k\,(2n+\gamma+1)_k\,k!}. \tag{2·2}$$

We prove (2·1) by induction on p and q using the Laplace transform techniques discussed in (3). Notice that by (1·9), (2·1) is equivalent to

$$z^k = \frac{(b_q)_k}{(a_p)_k} \sum_{n=0}^{k} \frac{(-k)_n (2n+\gamma)}{(n+\gamma+1)_k (n+\gamma)\, n!} {}_{p+2}F_q\left(\begin{matrix} -n, n+\gamma, a_p \\ b_q \end{matrix}\middle| z\right). \tag{2·3}$$

First we prove (2·3) for the case $p = q = 0$. It is known classically ((2), § 10·20) or deducible from ((3), (2·5)) that

$$z^k = (\alpha+1)_k \sum_{n=0}^{k} \frac{(-k)_n (2n+\alpha+\beta+1) P_n^{(\alpha,\beta)}(1-2z)}{(\alpha+1)_n (n+\alpha+\beta+2)_k (n+\alpha+\beta+1)}, \tag{2·4}$$

where
$$P_n^{(\alpha,\beta)}(x) = \frac{(-)^n (\beta+1)_n}{n!} {}_2F_1\left(\begin{matrix} -n, n+\alpha+\beta+1 \\ \beta+1 \end{matrix}\middle| \frac{1+x}{2}\right). \tag{2·5}$$

Making use of the property

$$P_n^{(\alpha,\beta)}(-x) = (-)^n P_n^{(\beta,\alpha)}(x), \tag{2·6}$$

interchanging α and β, and letting $\gamma = \alpha+\beta+1$, $b_1 = \beta+1$, we arrive at (2·3) for $p = 0$, $q = 1$. The case $p = q = 0$ follows simply from this case if we replace z by zb_1, divide both sides of (2·3) by b_1^k and let $b_1 \to \infty$.

Now assume (2·3) true for p and q. Multiplying both sides of (2·3) by $z^{\sigma-1}$, and taking the Laplace transform of both sides with respect to z, we obtain, see (4), p. 219, (17),

$$\begin{aligned}\int_0^\infty e^{-\lambda z} z^{\sigma-1+k}\, dz &= \frac{\Gamma(\sigma+k)}{\lambda^{\sigma+k}} \\ &= \frac{\Gamma(\sigma)(b_q)_k}{\lambda^\sigma (a_p)_k} \sum_{n=0}^{k} \frac{(-k)_n (2n+\gamma)}{(n+\gamma+1)_k (n+\gamma)\, n!} {}_{p+3}F_q\left(\begin{matrix} -n, n+\gamma, a_p, \sigma \\ b_q \end{matrix}\middle| \frac{1}{\lambda}\right).\end{aligned} \tag{2·7}$$

Replacing $1/\lambda$ by z completes the induction with respect to p. The induction with respect to q is effected by multiplying both sides of (2·3) by z^σ, letting $z = 1/\lambda$ and applying the inverse-Laplace transform, see (4), p. 297, (1). Note that for convergence of the integral in (2·7), the hypothesis $\Re(\sigma) > 0$ is necessary. By an appeal to the analytic continuation of (2·3), we may relax this restriction.

If in (2·3) we replace z by z/γ and let $\gamma \to \infty$, we arrive at the following

COROLLARY. *Let neither a_i nor b_j be a negative integer or zero. Then*

$$\pi_{k,n} = \frac{(b_q)_k (-k)_n}{(a_p)_k\, n!}, \tag{2·8}$$

and hence
$$C_n(\lambda) = \frac{(b_q)_n (-\lambda)^n}{(a_p)_n} \sum_{k=0}^{\infty} \frac{(n+b_q)_k (n+1)_k \xi_{k+n} \lambda^k}{(n+a_p)_k\, k!}. \tag{2·9}$$

3. *The shifted Jacobi polynomials.* The important case $p = 0$, $q = 1$, $\gamma = 1+\alpha+\beta$, $b_1 = 1+\beta$ of (2·2) and (1·9) yields†

$$\left.\begin{aligned} f(\lambda z) &= \sum_{n=0}^{\infty} S_n(\lambda) R_n^{(\alpha,\beta)}(z), \\ S_n(\lambda) &= \frac{n!\,\lambda^n}{(n+\alpha+\beta+1)_n} \sum_{k=0}^{\infty} \frac{(n+\beta+1)_k (n+1)_k}{(2n+\alpha+\beta+2)_k} \xi_{k+n} \frac{\lambda^k}{k!}, \end{aligned}\right\} \tag{3·1}$$

$$R_n^{(\alpha,\beta)}(z) = P_n^{(\alpha,\beta)}(2z-1). \tag{3·2}$$

† For $m = 0$, $(m+\alpha+\beta+1)_m = 1$, regardless of the values of α and β. This also applies to (3·5).

$R_n^{(\alpha,\beta)}(z)$ is called the shifted Jacobi polynomial and is interpolatory for z real, $0 \leqslant z \leqslant 1$. Using (3·1) with $\beta = \mp\frac{1}{2}$ together with the transformations ((5), 4·1·5)

$$R_n^{(\alpha,-\frac{1}{2})}(z^2) = \frac{(n+1)_n}{(n+\alpha+1)_n} P_{2n}^{(\alpha,\alpha)}(z), \tag{3·3}$$

$$zR_n^{(\alpha,\frac{1}{2})}(z^2) = \frac{(2n+1)\,(n+1)_n}{(2n+\alpha+1)\,(n+\alpha+1)_n} P_{2n+1}^{(\alpha,\alpha)}(z), \tag{3·4}$$

we can write

$$\left.\begin{aligned} f(\lambda z^2) &= \sum_{n=0}^{\infty} E_n(\lambda)\, P_{2n}^{(\alpha,\alpha)}(z), \\ E_n(\lambda) &= \frac{(2n)!\,(4\lambda)^n}{(2n+2\alpha+1)_{2n}} \sum_{k=0}^{\infty} \frac{(n+\frac{1}{2})_k\,(n+1)_k\,\xi_{k+n}\lambda^k}{(2n+\alpha+\frac{3}{2})_k\,k!}, \end{aligned}\right\} \tag{3·5}$$

$$\left.\begin{aligned} zf(\lambda z^2) &= \sum_{n=0}^{\infty} O_n(\lambda)\, P_{2n+1}^{(\alpha,\alpha)}(z), \\ O_n(\lambda) &= \frac{2(2n+1)!\,(4\lambda)^n}{(2n+2\alpha+2)_{2n+1}} \sum_{k=0}^{\infty} \frac{(n+\frac{3}{2})_k\,(n+1)_k\,\xi_{k+n}\lambda^k}{(2n+\alpha+\frac{5}{2})_k\,k!}. \end{aligned}\right\} \tag{3·6}$$

One special case of the foregoing is of particular interest. If $\alpha = \beta = -\frac{1}{2}$, $P_n^{(\alpha,\beta)}(z)$ is essentially the classical Chebyshev polynomial $T_n(z)$ which has proved so useful in approximation theory (6).

$$\left.\begin{aligned} T_n(z) &= \frac{n!}{(\frac{1}{2})_n} P_n^{(-\frac{1}{2},-\frac{1}{2})}(z), \\ &= T_n^*\left(\frac{1+z}{2}\right). \end{aligned}\right\} \tag{3·7}$$

$T_n^*(z)$ is called the shifted Chebyshev polynomial. Combining (3·5) with the quadratic transformation (6)

$$T_n^*(z^2) = T_{2n}(z), \tag{3·8}$$

one obtains

$$\left.\begin{aligned} f(\lambda z) &= \sum_{n=0}^{\infty} h_n(\lambda)\, T_n^*(z), \\ h_0(\lambda) &= \sum_{k=0}^{\infty} \frac{(\frac{1}{2})_k}{k!} \xi_k \lambda^k, \\ h_n(\lambda) &= 2(\tfrac{1}{4}\lambda)^n \sum_{k=0}^{\infty} \frac{(2n+k+1)_k}{k!} \xi_{k+n} (\tfrac{1}{4}\lambda)^k \quad (n \geqslant 1). \end{aligned}\right\} \tag{3·9}$$

If R is the radius of convergence of $f(z)$, then $S_n(\lambda)$, $E_n(\lambda)$, $O_n(\lambda)$ and $h_n(\lambda)$ converge for $|\lambda| < R$. Furthermore, $h_n(\lambda)$ is numerically small compared to the corresponding term of the Taylor series for $f(z)$; e.g. if $R > 1$, then $h_n(1)$ is of order $2^{1-2n}.f^{(n)}(0)/n!$.

In § 4 we demonstrate the usefulness of the above basic series by obtaining rapidly convergent approximations to some transcendental functions.

4. *Applications.* Our first example is the incomplete gamma function

$$\gamma(a+1, \lambda z) = \int_0^{\lambda z} e^{-t} t^a \, dt = \frac{(\lambda z)^{a+1}}{(a+1)} {}_1F_1\left(\begin{matrix} a+1 \\ a+2 \end{matrix}\middle| -\lambda z\right) \quad (a > -1). \tag{4·1}$$

In (3·1), let $\alpha = \beta = a$ and

$$\left.\begin{aligned} f(\lambda z) &= \lambda z\, {}_1F_1\left(\begin{matrix} a+1 \\ a+2 \end{matrix}\middle| -\lambda z\right), \\ \xi_k &= \frac{(a+1)\,\Gamma(a+k)\,(-)^{k-1}}{\Gamma(a+k+1)\,\Gamma(k)}. \end{aligned}\right\} \tag{4·2}$$

Then we find that

$$S_n(\lambda) = \frac{n(a+1)\,(-)^{n+1}\,\lambda^n}{(n+a)\,(n+2a+1)_n}\, {}_2F_2\left(\begin{matrix} n+1, n+a \\ n, 2n+2a+2 \end{matrix}\middle| -\lambda\right). \tag{4·3}$$

Now the equation

$${}_2F_2\left(\begin{matrix} n+1, n+a \\ n, 2n+2a+2 \end{matrix}\middle| -\lambda\right) = A\, {}_1F_1\left(\begin{matrix} n+a \\ 2n+2a \end{matrix}\middle| -\lambda\right) + (B+C\lambda)\, {}_1F_1\left(\begin{matrix} n+a+1 \\ 2n+2a+2 \end{matrix}\middle| -\lambda\right) \tag{4·4}$$

may be solved for A, B and C to yield

$$A = -\frac{a(2n+2a+1)}{n(n+a+1)}, \quad B = 1 + \frac{a(2n+2a+1)}{n(n+a+1)}, \quad C = \frac{-a}{2n(n+a+1)}. \tag{4·5}$$

Converting the confluent hypergeometric functions in (4·4) into Bessel functions by (2), §7·2·2, (12) and referring to (4·1), we have the expansion

$$\begin{aligned} \gamma(a+1, \lambda z) &= z^a(\pi\lambda)^{\frac{1}{2}}\, e^{-\frac{1}{2}\lambda} \sum_{n=0}^{\infty} \frac{(-)^n\,(n+a+1)_a\,(n+a+\frac{1}{2})}{(n+a)\,(n+a+1)} \\ &\times \left\{ a I_{n+a-\frac{1}{2}}(\tfrac{1}{2}\lambda) + \left[a - \frac{2(n+a)\,(n+2a+1)}{\lambda}\right] I_{n+a+\frac{1}{2}}(\tfrac{1}{2}\lambda) \right\} R_n^{(a,a)}(z) \quad (a > -1). \end{aligned} \tag{4·6}$$

If $z = 1$ in (4·6), the result agrees with an expansion given by Buchholz ((7), p. 130).

Our second example, the error function, is a special case of (4·1) but here the expansion is of a different nature. Let

$$z f(\lambda z^2) = z\, {}_1F_1\left(\begin{matrix} 1 \\ \frac{3}{2} \end{matrix}\middle| a^2 z^2\right). \tag{4·7}$$

Then from (3·6) with $\alpha = -\frac{1}{2}$, $\lambda = a^2$ and from (2), §9·9, (1) we find that

$$\operatorname{erf}(z) = \int_0^z e^{-t^2} dt = \sqrt{\pi}\, e^{\frac{1}{2}a^2 - z^2} \sum_{n=0}^{\infty} I_{n+\frac{1}{2}}(\tfrac{1}{2}a^2)\, T_{2n+1}(z/a). \tag{4·8}$$

Let ϵ_N denote the maximum error incurred when just N terms of (4·8) are retained. Use of (2), §10·18 and (8), p. 49, no. (1) provides the estimate

$$|\epsilon_N| \leqslant \frac{\sqrt{\pi}\, e^{a^2 - z^2} |\,\frac{1}{2}a\,|^{2N+1}}{\Gamma(N+\frac{3}{2})} |1 + O(1/N)| \quad (a \text{ real}, \; -a \leqslant z \leqslant a). \tag{4·9}$$

If $a = z$ or $a^2 = 2z^2$ in (4·8) expansions given by Luke and Coleman (9) and Tricomi (10), respectively, result.

More generally, if f is hypergeometric in nature, the Theorem and Corollary in § 3 yield many expansions given previously by the authors (3). Also, if only the numerical values of the coefficients for the Taylor's series of $f(z)$ are known, the formulae in §§ 2 and 3 are valuable. In the accompanying table are listed coefficients for the Chebyshev polynomial expansions of the Riemann zeta function, the gamma function, and its reciprocal. These were obtained by applying (3·9) to the tabulations (11), (12) and (13), p. 4. For (12), the authors thank J. C. P. Miller who graciously loaned us his manuscript. The coefficients in the table are given to a sufficient number of decimals to minimize round-off error in case the expansions are rearranged in powers of z. The number in parentheses behind each entry is the power of ten by which the entry is to be multiplied.

$$\zeta(1+z) = \frac{1}{z} + \sum_{n=0}^{6} h_n T_n^*(z) + \eta(z), \quad \Gamma(z+3) = \sum_{n=0}^{10} h_n T_n^*(z) + \eta(z), \quad \frac{1}{\Gamma(z)} = z \sum_{n=0}^{10} h_n T_n^*(z) + \eta(z),$$

$$|\eta(z)| \leqslant 10^{-10} \quad (0 < z \leqslant 1) \qquad |\eta(z)| \leqslant 10^{-10} \quad (0 \leqslant z \leqslant 1) \qquad |\eta(z)| \leqslant 10^{-10} \quad (0 \leqslant z \leqslant 1)$$

n	h_n		n	h_n		n	h_n	
0	0·61172 44860	(+0)	0	0·36573 87725 10	(+1)	0	0·10637 73007 8	(+1)
1	0·33864 46129	(−1)	1	0·19575 43456 67	(+1)	1	−0·49855 8735	(−2)
2	−0·65024 13912	(−3)	2	0·33829 71138 1	(+0)	2	−0·64192 54364	(−1)
3	−0·52464 ~~23167~~	(−5)	3	0·42089 51276 7	(−1)	3	0·50657 9862	(−2)
4	0·62132 10656	(−6)	4	0·42876 50483	(−2)	4	0·41660 9128	(−3)
5	−0·13838 69984	(−7)	5	0·36521 21696	(−3)	5	−0·80481 417	(−4)
6	−0·14078 641	(−10)	6	0·27400 6424	(−4)	6	0·29600 098	(−5)
			7	0·18124 024	(−5)	7	0·26897 5	(−6)
			8	0·10965 78	(−6)	8	−0·33397	(−7)
			9	0·59871 9	(−8)	9	0·10896	(−8)
			10	0·30769 3	(−9)	10	0·514	(−10)

For computational purposes, the expansions need not be rearranged in powers of z. Clenshaw (14), basing his work on the recursion formulae for $T_n^*(z)$, has developed a nesting procedure to utilize the coefficients in such tables without computing $T^*(z)$ or powers of z.

For the general case, $G_n(z, \gamma)$ itself can always be computed recursively. If we employ the techniques in Rainville (15), we can show that $G_n(z, \gamma)$ obeys a linear recursion relation of the form

$$\left.\begin{aligned} &\sum_{j=0}^{s} (A_j + zB_j)\, G_{n-j}(z, \gamma) = 0, \\ &B_0 = B_s = 0, \quad A_0 = 1, \end{aligned}\right\} \tag{4·10}$$

where $s = \max\{p+2, q+1\}$. For $s = 3$, the authors have calculated the A_j and B_j in (4·10), while for $s = 2$, the results are classical. For $s \geqslant 3$, the determination of the A_j and B_j become tedious and for such values of s no general results of the type (4·10) seem to exist in the literature. However, specializations of the authors' formulae for $s = 3$ are given in (15), p. 233 ff.

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Polynomial Expansions of Bessel Functions and Some Associated Functions

By Jet Wimp

1. Introduction. In this paper we first determine representations for the Anger-Weber functions $\mathbf{J}_v(ax)$ and $\mathbf{E}_v(ax)$ in series of symmetric Jacobi polynomials. (These include Legendre and Chebyshev polynomials as special cases.) If v is an integer, these become expansions for the Bessel function of the first kind, since $\mathbf{J}_k(ax) = J_k(ax)$. In Section 3, corresponding representations are found for $(ax)^{-v}J_v(ax)$. Convenient error bounds are obtained for the above expansions. In the fourth section we determine the similar type expansions for the Bessel functions $Y_k(ax)$ and $K_k(ax)$. In Section 5, the coefficients of some of our expansions are tabulated for particularly important values of the various parameters.

2. Symmetric Jacobi Expansions of Anger-Weber Functions. A function $f(x)$ satisfying certain conditions (for these consult [1]) may be expanded in the series

$$f(x) = \sum_{n=0}^{\infty} C_n P_n^{(\alpha,\alpha)}(x), \quad -1 \leqq x \leqq 1, \quad \alpha > -1, \tag{2.1}$$

where $P_n^{(\alpha,\alpha)}(x)$ is called the symmetric Jacobi polynomial of degree n. For our present purposes we shall use a definition given in [2]:

$$2^n n! P_n^{(\alpha,\alpha)}(x) = (-)^n(1 - x^2)^{-\alpha}D^n[(1 - x^2)^{\alpha+n}]. \tag{2.2}$$

Also

$$C_n = h_n^{-1} \int_{-1}^{1} f(x)(1 - x^2)^\alpha P_n^{(\alpha,\alpha)}(x)\, dx, \tag{2.3}$$

$$h_n = \frac{2^{2\alpha}(n + 1)_\alpha}{(n + \alpha + \frac{1}{2})(n + \alpha + 1)_\alpha}; \qquad (v)_\mu = \frac{\Gamma(v + \mu)}{\Gamma(v)}, \qquad (v)_0 = 1. \tag{2.4}$$

Using the representation (2.2) in (2.3) and noticing that all derivatives of $(1 - x^2)^{\alpha+n}$ up to and including the $(n - 1)$st vanish at $x = \pm 1$, we integrate (2.3) n times by parts to get:

$$C_n = (2^n n!\, h_n)^{-1} \int_{-1}^{1} f^{(n)}(x)(1 - x^2)^{\alpha+n}\, dx. \tag{2.5}$$

Consider the integral definition of the Anger-Weber functions [2, v. 2, p. 35]

$$\mathbf{J}_v(ax) + i\mathbf{E}_v(ax) = \pi^{-1} \int_0^\pi e^{i[v\phi - ax \sin \phi]}\, d\phi = f(x). \tag{2.6}$$

When v is an integer, $\mathbf{J}_v(ax)$ coincides with the Bessel function of the first kind $J_v(ax)$ [2, v. 2, p. 4].

Now differentiate (2.6) n times under the integral sign, substitute the result in

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(2.5) and interchange the order of integration (which is, of course, permissible). The inner integral is known [3] and after evaluating it we have

$$C_n = (-i)^n(n+1)_\alpha(h_n\pi^{1/2})^{-1} \cdot \int_0^\pi e^{iv\phi}\left\{\frac{a \sin\phi}{2}\right\}^{-[\alpha+(1/2)]} J_{[n+\alpha+(1/2)]}(a \sin\phi)\, d\phi. \tag{2.7}$$

Use the power series expansion for the Bessel function in (2.7) and integrate term-by-term to get

$$C_n = (-i)^n\left[\cos\frac{v\pi}{2} + i \sin\frac{v\pi}{2}\right]\Lambda_n R_n(v, \alpha, a), \tag{2.8}$$

where

$$\Lambda_n = \frac{a^n n!}{\Gamma\left(\frac{n}{2}+\frac{v}{2}+1\right)\Gamma\left(\frac{n}{2}-\frac{v}{2}+1\right)(n+2\alpha+1)_n}, \tag{2.9}$$

and R_n is conveniently described in hypergeometric notation [2, v. 1, p. 182] as

$$R_n(v, \alpha, a) = {}_2F_3\left[\frac{n}{2}+\frac{1}{2}, \frac{n}{2}+1; \alpha+n+\frac{3}{2}, \frac{n}{2}+\frac{v}{2}+1, \frac{n}{2}-\frac{v}{2}+1; -\frac{a^2}{4}\right] \tag{2.10}$$

Equating real and imaginary parts of (2.6) and (2.1), we get

$$\mathbf{J}_v(ax) = \sum_{n=0}^{\infty} A_n P_n^{(\alpha,\alpha)}(x), \quad -1 \leqq x \leqq 1, \tag{2.11}$$

$$\mathbf{E}_v(ax) = \sum_{n=0}^{\infty} B_n P_n^{(\alpha,\alpha)}(x), \quad -1 \leqq x \leqq 1, \tag{2.12}$$

where

$$A_n = \Lambda_n R_n(v, \alpha, a)\phi_n(v), \tag{2.13}$$

$$B_n = \Lambda_n R_n(v, \alpha, a)\psi_n(v), \tag{2.14}$$

and

$$\phi_n(v) = \begin{cases} (-)^{n/2}\cos\frac{v\pi}{2}, & n \text{ even}, \\ (-)^{(n-1)/2}\sin\frac{v\pi}{2}, & n \text{ odd}; \end{cases} \tag{2.15}$$

$$\psi_n(v) = \begin{cases} (-)^{n/2}\sin\frac{v\pi}{2}, & n \text{ even}, \\ (-)^{(n+1)/2}\cos\frac{v\pi}{2}, & n \text{ odd}. \end{cases} \tag{2.16}$$

Equations (2.11) and (2.12) and the expansions in Section 3 may also be derived from results in [4]. The present derivation is more satisfactory because it establishes a foundation for the work in Section 4.

When $\alpha = -\frac{1}{2}$,

$$P_n^{[-(1/2),-(1/2)]}(x) = (\tfrac{1}{2})_n (n!)^{-1} T_n(x), \quad n = 1, 2 \cdots, \tag{2.17}$$

where $T_n(x)$ is the Chebyshev polynomial of the first kind of degree n. Also for this value of α, R_n simplifies to the product of two Bessel functions [2, v. 2, p. 11]. With $\alpha = -\frac{1}{2}$, then (2.11)–(2.14) become

$$\mathrm{J}_v(ax) = \sum_{n=0}^{\infty} C_n T_n(x), \quad -1 \leqq x \leqq 1, \tag{2.18}$$

$$\mathrm{E}_v(ax) = \sum_{n=0}^{\infty} D_n T_n(x), \quad -1 \leqq x \leqq 1, \tag{2.19}$$

where

$$C_n = \epsilon_n J_{(n+v)/2}\left(\frac{a}{2}\right) J_{(n-v)/2}\left(\frac{a}{2}\right) \phi_n(v), \tag{2.20}$$

$$D_n = \epsilon_n J_{(n+v)/2}\left(\frac{a}{2}\right) J_{(n-v)/2}\left(\frac{a}{2}\right) \psi_n(v), \tag{2.21}$$

and $\epsilon_n = \begin{cases} 1, & n = 0 \\ 2, & n > 0. \end{cases}$

For integral v we have the expansions

$$J_{2k}(ax) = \sum_{n=0}^{\infty} \epsilon_n J_{k+n}\left(\frac{a}{2}\right) J_{k-n}\left(\frac{a}{2}\right) T_{2n}(x), \quad -1 \leqq x \leqq 1, \tag{2.22}$$

$$J_{2k+1}(ax) = 2\sum_{n=0}^{\infty} J_{k+n+1}\left(\frac{a}{2}\right) J_{k-n}\left(\frac{a}{2}\right) T_{2n+1}(x), \quad -1 \leqq x \leqq 1, \tag{2.23}$$

and $k = 0, 1, 2 \cdots$. Equation (2.22) is known [2, v. 2, p. 100].

Since

$$J_v(iz) = e^{(v\pi i)/2} I_v(z), \tag{2.24}$$

where $I_v(z)$ is the modified Bessel function of the first kind [2, v. 2, p. 5], we may replace a by ia in (2.22) and (2.23) to get expansions for $I_{2k}(ax)$ and $I_{2k+1}(ax)$.

It is important to note that, although the above expansions are valid only for x real and $|x| \leqq 1$, (2.6) is entire in a and v, and hence a may be chosen arbitrarily to yield expansions valid anywhere in the finite complex plane.

The expansions (2.11), (2.12), (2.18), (2.19), (2.22), and (2.23) are quite rapidly convergent, particularly in the Chebyshev cases [5]; consequently the last four expansions are eminently suitable for use on digital computers.* Such series

* The Bessel functions required to compute the coefficients in our expansions can be systematically generated on electronic computers with the aid of techniques discussed in [6, 7, 8]. There are numerous tables available for hand calculations. The words "accuracy," "error," and "convergence" in this paper always refer to the properties of the expansion when truncated after a finite number of terms.

may be truncated and rearranged in powers of x. Clenshaw [9], though, by using the recursion formulas satisfied by the Chebyshev polynomials, has formulated a convenient nesting procedure which allows one to utilize such expansions directly. The scheme is as follows. Consider

$$f^{(1)}(x) = \sum_{n=0}^{N} A_n^{(1)} T_n^* \left(\frac{x}{a}\right), \quad 0 \leqq x \leqq a, \tag{2.25}$$

$$f^{(2)}(x) = \sum_{n=0}^{N} A_n^{(2)} T_{2n} \left(\frac{x}{a}\right), \quad -a \leqq x \leqq a, \tag{2.26}$$

$$f^{(3)}(x) = \sum_{n=0}^{N} A_n^{(3)} T_{2n+1} \left(\frac{x}{a}\right), \quad -a \leqq x \leqq a. \tag{2.27}$$

To evaluate the series (2.25), (2.26), or (2.27), respectively, we construct the following sequences:

$$b_n^{(1)} = \left[4\left(\frac{x}{a}\right) - 2\right] b_{n+1}^{(1)} - b_{n+2}^{(1)} + A_n^{(1)}, \tag{2.28}$$

$$b_n^{(2)} = \left[4\left(\frac{x}{a}\right)^2 - 2\right] b_{n+1}^{(2)} - b_{n+2}^{(2)} + A_n^{(2)}, \tag{2.29}$$

$$b_n^{(3)} = \left[4\left(\frac{x}{a}\right)^2 - 2\right] b_{n+1}^{(3)} - b_{n+2}^{(3)} + A_n^{(3)}, \tag{2.30}$$

for $n = N, N-1, N-2, \cdots, 3, 2, 1, 0$ with the initial values

$$b_{N+1}^{(1)} = b_{N+2}^{(1)} = b_{N+1}^{(2)} = b_{N+2}^{(2)} = b_{N+1}^{(3)} = b_{N+2}^{(3)} = 0.$$

$f^{(1)}(x), f^{(2)}(x)$, and $f^{(3)}(x)$ are then given by

$$f^{(1)}(x) = b_0^{(1)} + b_1^{(1)} \left[1 - 2\left(\frac{x}{a}\right)\right], \tag{2.31}$$

$$f^{(2)}(x) = b_0^{(2)} + b_1^{(2)} \left[1 - 2\left(\frac{x}{a}\right)^2\right], \tag{2.32}$$

$$f^{(3)}(x) = [b_0^{(3)} - b_1^{(3)}]\left(\frac{x}{a}\right). \tag{2.33}$$

The method is as direct as the ordinary nesting process used to evaluate polynomials.

We now derive error estimates for the expansions (2.11) and (2.12) for $-1 \leqq x \leqq 1$. Notice that

$$R_n(v, \alpha, a) = 1 + 0\left(\frac{1}{n}\right) \tag{2.34}$$

provided all other parameters are fixed, and consequently

$$|A_n| \leqq \frac{|a|^n n!}{\Gamma\left(\frac{n+v}{2}+1\right)\Gamma\left(\frac{n-v}{2}+1\right)(n+2\alpha+1)_n} \left|1 + 0\left(\frac{1}{n}\right)\right|, \tag{2.35}$$

and likewise for B_n. Also [2, v. 2, p. 206]

$$\max_{-1\leqq x\leqq 1} | P_n^{(\alpha,\alpha)}(x) | = \binom{n+\alpha}{n}, \quad \alpha \geqq -\tfrac{1}{2}. \tag{2.36}$$

Let ϵ_N denote the error incurred by taking just N terms of (2.11) or (2.12). Because of the rapidity of convergence of the expansions, as shown by (2.35), the $(N+1)$th term furnishes us with a convenient error estimate

$$| \epsilon_N | = \frac{| a |^N N^{\alpha+(1/2)} \pi^{1/2}}{2^{2N+2\alpha}\Gamma\left(\frac{N+v}{2}+1\right)\Gamma\left(\frac{N-v}{2}+1\right)\Gamma(\alpha+1)} \left| 1 + 0\left(\frac{1}{N}\right)\right|, \tag{2.37}$$

where $\alpha \geqq -\frac{1}{2}$, $N > v$, $-1 \leqq x \leqq 1$.

Among the values of α considered, it follows from (2.37) that the choice $\alpha = -\frac{1}{2}$, i.e., the Chebyshev case, yields the smallest error term for large N.

3. Expansions of Bessel Functions of the First Kind of Nonintegral Order. Results in the previous section gave symmetric Jacobi polynomial expansions for $J_v(ax)$ and $I_v(ax)$ for integral v. When v is nonintegral, these functions are no longer entire functions of x, and it is convenient to derive an expansion for the entire function

$$\Gamma(v+1)(ax/2)^{-v}J_v(ax) = {}_0F_1\left(v+1; -\frac{a^2x^2}{4}\right). \tag{3.1}$$

Corresponding expansions for $\Gamma(v+1)(ax/2)^{-v}I_v(ax)$ then follow, as before, from (2.24).

Let $f(x)$ in (2.5) be the right-hand side of (3.1). Then we have

$$J_v(ax) = (ax)^v \sum_{n=0}^{\infty} A_n P_{2n}^{(\alpha,\alpha)}(x), \quad -1 \leqq x \leqq 1, \tag{3.2}$$

where

$$\begin{aligned} A_n = {} & \frac{(-)^n(2a)^{2n}}{2^v\pi^{1/2}(2n+2\alpha+1)_{2n}(n+\frac{1}{2})_{v+(1/2)}} \\ & \cdot {}_1F_2\left(n+\frac{1}{2};\ v+n+1,\ 2n+\alpha+\frac{3}{2};\ -\frac{a^2}{4}\right). \end{aligned} \tag{3.3}$$

These equations also follow from a result in [4]. Indeed, using a general expansion given there, an alternative formula for (3.3) can be stated. We have

$${}_1F_2\left[\rho; \sigma, \tau; -\frac{z^2}{4}\right] = \Gamma(\sigma)(z/2)^{1-\sigma}\sum_{k=0}^{\infty}\frac{(z/2)^k(\tau-\rho)_k}{k!\,(\tau)_k}J_{k+\sigma-1}(z), \tag{3.4}$$

$$\begin{aligned} A_n = {} & \frac{(-)^n 2^{(1/2)-\alpha-v}\Gamma(n+\frac{1}{2})(2n+\alpha+\frac{1}{2})(2n+\alpha+1)_\alpha}{a^{(1/2)+\alpha}} \\ & \cdot \sum_{k=0}^{\infty}\frac{(a/2)^k(v+\frac{1}{2})_k}{k!\,\Gamma(v+n+k+1)} J_{2n+k+\alpha+(1/2)}(a). \end{aligned} \tag{3.5}$$

For the Chebyshev case of (3.2) $\alpha = -\frac{1}{2}$ and

$$J_v(ax) = (ax)^v \sum_{n=0}^{\infty} C_n T_{2n}(x), \quad -1 \leqq x \leqq 1, \tag{3.6}$$

where

$$C_n = \frac{\epsilon_n(-)^n(a/4)^{2n}}{2^v n!\,\Gamma(v+n+1)}\, {}_1F_2\left[n+\tfrac{1}{2};\ v+n+1, 2n+1;\ -\frac{a^2}{4}\right]. \tag{3.7}$$

Notice that when $v = -\frac{1}{2}$, (3.3) simplifies. Also, since

$$J_{-(1/2)}(ax) = \left(\frac{\pi ax}{2}\right)^{-(1/2)} \cos(ax), \tag{3.8}$$

we infer the expansion

$$\cos(ax) = \sum_{n=0}^{\infty} C_n P_{2n}^{(\alpha,\alpha)}(x), \quad -1 \leqq x \leqq 1, \tag{3.9}$$

where

$$C_n = \frac{(-)^n \pi^{1/2} 2^{-\alpha+(1/2)}(2n+\alpha+\frac{1}{2})(2n+\alpha+1)_\alpha}{a^{\alpha+(1/2)}} J_{2n+\alpha+(1/2)}(a), \tag{3.10}$$

a formula which can be derived in a number of different ways.

Using an analysis similar to that of Section 2, we may derive the estimate for the error incurred when just N terms of (3.2) are used.

$$|\epsilon_N| = \frac{|a|^{v+2N}\,|x|^v\,\pi^{1/2}N^{\alpha+(1/2)}}{2^{4N+\alpha+v-(1/2)}N!\,\Gamma(N+v+1)\Gamma(\alpha+1)}\left|1+0\left(\frac{1}{N}\right)\right|, \tag{3.11}$$

$$-1 \leqq x \leqq 1, \quad \alpha \geqq -\tfrac{1}{2}, \quad N > v.$$

Concerning the optimum choice of α in (3.2), see the discussion surrounding (2.37).

4. Expansions of Bessel Functions of the Second Kind. The Bessel function and modified Bessel function of the second kind are denoted by $Y_v(z)$ and $K_v(z)$, respectively, and a treatment of them can be found in [2, v. 2, Ch. VII]. If v is non-integral, then

$$Y_v(z) = [\sin(v\pi)]^{-1}\{J_v(z)\cos(v\pi) - J_{-v}(z)\}, \tag{4.1}$$

and

$$K_v(z) = \frac{\pi}{2}[\sin(v\pi)]^{-1}\{I_{-v}(z) - I_v(z)\}, \tag{4.2}$$

so for such values of v expansions for the functions follow directly from the results of Section 3.

If v is an integer, it can be shown that

$$Y_k(ax) = \frac{2}{\pi}\left[\gamma + \ln\left(\frac{ax}{2}\right)\right]J_k(ax) + N_{k-1}(ax) - \frac{1}{\pi}W_k(ax), \tag{4.3}$$

and

$$(4.4)\quad K_k(ax) = (-)^{k+1}\left[\gamma + \ln\left(\frac{ax}{2}\right)\right] I_k(ax) - \frac{\pi}{2}\, i^k N_{k-1}(iax) + \frac{i^k}{2} W_k(iax),$$

where

$$(4.5)\quad N_{k-1}(ax) = \begin{cases} -\dfrac{1}{\pi}\displaystyle\sum_{m=0}^{k-1}\left(\frac{ax}{2}\right)^{2m-k}\frac{(k-m-1)!}{m!}, & k > 0 \\ 0, & k = 0, \end{cases}$$

and

$$(4.6)\quad W_k(ax) = \sum_{m=0}^{\infty}(-)^m\left(\frac{ax}{2}\right)^{k+2m}\frac{[h_{m+k} + h_m]}{m!(k+m)!}.$$

In the above $\gamma = 0.57721 \cdots =$ Euler's constant and

$$(4.7)\quad h_m = 1 + \tfrac{1}{2} + \cdots + \frac{1}{m}, \quad h_0 = 0$$

We assume the value of log $(ax/2)$ is known. Then, since expansions for $J_k(ax)$ and $I_k(ax)$ were found in Section 2, and since $N_{k-1}(ax)$ is simply a polynomial in $1/(ax)$, we need expand only the entire part of (4.3), i.e., $W_k(ax)$, in symmetric Jacobi polynomials.

Using the representation (4.6) as $f(x)$ in formula (2.5), a straight-forward derivation gives the series

$$(4.8)\quad W_k(ax) = \sum_{n=0}^{\infty} A_n P_n^{(\alpha,\alpha)}(x), \quad -1 \leqq x \leqq 1,$$

where

$$(4.9)\quad \begin{aligned} A_n = {} & \frac{[(-)^k + (-)^n](n+\alpha+1)_\alpha(n+\alpha+\frac{1}{2})}{2^{n+2\alpha+1}} \\ & \cdot\sum_{m=0}^{\infty}\frac{(-)^m(-k-2m)_n}{\left(m+\dfrac{k-n+1}{2}\right)_{n+\alpha+1}}\left(\frac{a}{2}\right)^{k+2m}\frac{[h_{k+m}+h_m]}{m!(k+m)!}. \end{aligned}$$

We note that the expansion for $Y_0(ax)$ may also be obtained by partially differentiating (3.2) with respect to v since

$$(4.10)\quad Y_0(ax) = 2\pi^{-1}\left\{\frac{\partial J_v(ax)}{\partial v}\right\}_{v=0}.$$

A similar procedure yields the expansion for $K_0(ax)$. The Jacobi series for $Y_k(ax)$ and $K_k(ax)$ for $k > 0$, however, are not so easily obtained in this manner.

For $k = 0$ and 1, the Chebyshev cases of (4.3) and (4.4) are

$$(4.11)\quad Y_0(ax) = \frac{2}{\pi}\left[\gamma + \ln\left(\frac{ax}{2}\right)\right] J_0(ax) + \sum_{n=0}^{\infty} E_n\, T_{2n}(x), \qquad 0 < x \leqq 1,$$

$$(4.12)\quad Y_1(ax) = \frac{2}{\pi}\left[\gamma + \ln\left(\frac{ax}{2}\right)\right] J_1(ax) - \frac{2}{\pi ax} + \sum_{n=0}^{\infty} F_n\, T_{2n+1}(x), \quad 0 < x \leqq 1,$$

$$(4.13)\quad K_0(ax) = -\left[\gamma + \ln\left(\frac{ax}{2}\right)\right] I_0(ax) + \sum_{n=0}^{\infty} G_n\, T_{2n}(x), \qquad 0 < x \leqq 1,$$

$$(4.14)\quad K_1(ax) = \left[\gamma + \ln\left(\frac{ax}{2}\right)\right] I_1(ax) + \frac{1}{ax} + \sum_{n=0}^{\infty} H_n\, T_{2n+1}(x), \qquad 0 < x \leqq 1,$$

where

$$(4.15)\quad E_n = \frac{2\epsilon_n \left(\frac{a}{4}\right)^{2n} (-)^{n+1}}{\pi (n!)^2} \sum_{k=0}^{\infty} \frac{(-)^k \left(\frac{a}{2}\right)^{2k} \left(n + \frac{1}{2}\right)_k h_{n+k}}{(n+1)_k (2n+1)_k\, k!},$$

$$(4.16)\quad F_n = \frac{2(-)^{n+1} \left(\frac{a}{4}\right)^{2n+1}}{\pi n!(n+1)!} \sum_{k=0}^{\infty} \frac{(-)^k \left(\frac{a}{2}\right)^{2k} \left(n + \frac{3}{2}\right)_k [h_{n+k+1} + h_{n+k}]}{(n+2)_k (2n+2)_k\, k!},$$

$$(4.17)\quad G_n = \frac{\epsilon_n \left(\frac{a}{4}\right)^{2n}}{(n!)^2} \sum_{k=0}^{\infty} \frac{\left(\frac{a}{2}\right)^{2k} \left(n + \frac{1}{2}\right)_k h_{n+k}}{(n+1)_k (2n+1)_k\, k!},$$

$$(4.18)\quad H_n = -\frac{\left(\frac{a}{4}\right)^{2n+1}}{n!(n+1)!} \sum_{k=0}^{\infty} \frac{\left(\frac{a}{2}\right)^{2k} \left(n + \frac{3}{2}\right)_k [h_{n+k+1} + h_{n+k}]}{(n+2)_k (2n+2)_k\, k!}.$$

5. Tables. Tables 1 through 3 are based on the Chebyshev polynomial cases of the expansions given in the previous sections of this paper. The entries in Tables 1 and 2 were computed on the UNIVAC 1103-A and those in Table 3 on the IBM 7090 at ASD. The calculations were designed so that the error incurred in using the expansions whose coefficients are tabulated here will not exceed five units in the 15th decimal place. Spot checks indicate the error is even less. Because all entries are to 16 significant figures, the expansions may be rearranged in powers of x with no loss of accuracy.

The number in parentheses after each entry is the power of ten by which the entry is to be multiplied. We have chosen coefficients corresponding to $a = 5$, but the coefficients for other values of a from one through ten are available on request.

Note that the expansions in this paper are valid not only for $-1 \leqq x \leqq 1$ but for complex x in a region which can be determined by a theorem of Szegö [1, p. 238]. More specifically, a Jacobi series representing an entire function converges everywhere in the finite complex plane. However, the further x lies away from $-1 \leqq x \leqq 1$, the more the accuracy of the expansion deteriorates. This is so because $P_n^{(\alpha,\alpha)}(x)$ for complex x can no longer be bounded by a simple power of n but behaves in the following manner [10]

$$(5.1)\quad P_n^{(\alpha,\alpha)}(z) = \frac{\Gamma(n+\alpha+1)}{n!\pi^{1/2}} N^{2\gamma} \left(\sin\frac{\theta}{2}\right)^{2\gamma} \left(\cos\frac{\theta}{2}\right)^{-2\gamma-2\alpha-1} \cdot \cos\left[N\theta + \pi\gamma\right] \left\{1 + 0\left(\frac{1}{N}\right)\right\}$$

valid in the z plane cut from -1 to $-\infty$ and from 1 to ∞. In (5.1), $\cos\theta = z$,

TABLE 1

Coefficients for the Series

$$J_0(x) = \sum_{n=0}^{\infty} A_n T_{2n}(x/5) \qquad J_1(x) = \sum_{n=0}^{\infty} B_n T_{2n+1}(x/5) \qquad I_0(x) = \sum_{n=0}^{\infty} C_n T_{2n}(x/5) \qquad I_1(x) = \sum_{n=0}^{\infty} D_n T_{2n+1}(x/5)$$

$$-5 \leqq x \leqq 5$$

n	A_n	B_n	C_n	D_n
0	2.34098 98253 24576 (−03)	−4.81025 79874 58212 (−02)	1.08230 41593 72444 (+01)	1.65591 83236 43522 (+01)
1	−4.94205 09340 95238 (−01)	−4.43466 65460 22008 (−01)	1.26677 21318 60009 (+01)	6.42500 61815 55151 (+00)
2	3.97937 36723 20755 (−01)	1.93233 13296 91198 (−01)	3.25873 16530 56593 (+00)	1.21103 55366 64603 (+00)
3	−9.38314 58796 59383 (−02)	−3.19623 68142 70534 (−02)	4.50054 56944 84814 (−01)	1.30904 56998 82480 (−01)
4	1.08875 31648 68103 (−02)	2.87773 31330 73935 (−03)	3.80753 97089 75161 (−02)	9.06329 93010 42887 (−03)
5	−7.60626 76577 33766 (−04)	−1.64773 93001 95709 (−04)	2.15738 77227 50612 (−03)	4.33748 41004 04394 (−04)
6	3.56948 36463 56946 (−05)	6.56128 50055 62524 (−06)	8.72062 45418 29624 (−05)	1.51584 32032 61766 (−05)
7	−1.20606 97061 36835 (−06)	−1.92705 34880 37504 (−07)	2.63488 07999 39597 (−06)	4.03099 55295 59207 (−07)
8	3.07903 85720 34403 (−08)	4.35311 98064 51247 (−09)	6.16685 39084 19237 (−08)	8.42090 28170 89711 (−09)
9	−6.15440 55412 06848 (−10)	−7.80521 83217 68401 (−11)	1.14988 29923 31926 (−09)	1.41745 27229 98507 (−10)
10	9.89883 30623 59870 (−12)	1.13848 12811 94913 (−12)	1.74728 40587 55705 (−11)	1.96254 75993 94277 (−12)
11	−1.30938 62877 22900 (−13)	−1.37786 52001 23884 (−14)	2.20433 13796 56194 (−13)	2.27359 85296 82644 (−14)
12	1.44992 54555 46092 (−15)		2.34504 22753 50325 (−15)	

TABLE 2

Coefficients for the Series

$$Y_0(x) = (2/\pi)\{\gamma + ln(x/2)\}J_0(x) + \sum_{n=0}^{\infty} E_n T_{2n}(x/5) \qquad Y_1(x) = (2/\pi)\{\gamma + ln(x/2)\}J_1(x) + \sum_{n=0}^{\infty} F_n T_{2n+1}(x/5) - 2/\pi x$$

$$K_0(x) = -\{\gamma + ln(x/2)\}I_0(x) + \sum_{n=0}^{\infty} G_n T_{2n}(x/5) \qquad K_1(x) = \{\gamma + ln(x/2)\}I_1(x) + \sum_{n=0}^{\infty} H_n T_{2n+1}(x/5) + 1/x$$

$$0 < x \leqq 5$$

n	E_n	F_n	G_n	H_n
0	2.06225 35144 48362 (−01)	6.16723 32064 62446 (−01)	1.44570 70580 38540 (+01)	−2.33572 75478 64823 (+01)
1	−1.66649 68241 18285 (−01)	1.01148 30681 23224 (−01)	1.94363 50195 81009 (+01)	−1.05995 40069 53968 (+01)
2	−2.62525 23404 65465 (−01)	−1.63819 01170 12895 (−01)	5.80235 95441 72479 (+00)	−2.28801 59175 41593 (+00)
3	9.57577 15009 74545 (−02)	3.61543 51643 46677 (−02)	9.01848 87443 62640 (−01)	−2.74798 33365 99269 (−01)
4	−1.34756 13306 42697 (−02)	−3.78893 02501 89891 (−03)	8.36577 27959 36959 (−02)	−2.06655 58368 06775 (−02)
5	1.06032 54643 99260 (−03)	2.39870 11377 45682 (−04)	5.10092 82215 62887 (−03)	−1.05709 50504 93099 (−03)
6	−5.41408 18903 44878 (−05)	−1.02859 55207 54039 (−05)	2.18934 64152 72086 (−04)	−3.90317 93418 52240 (−05)
7	1.95165 72417 48920 (−06)	3.20202 24989 85112 (−07)	6.95536 19902 36087 (−06)	−1.08728 98645 17578 (−06)
8	−5.24918 47596 77735 (−08)	−7.58631 71289 13763 (−09)	1.69908 13404 94038 (−07)	−2.36391 19626 27828 (−08)
9	1.09579 84960 92691 (−09)	1.41602 52452 47205 (−10)	3.28795 17709 40376 (−09)	−4.12016 28736 68884 (−10)
10	−1.82911 13098 12991 (−11)	−2.13823 11419 29658 (−12)	5.16184 89074 51723 (−11)	−5.88302 36091 76335 (−12)
11	2.49885 87269 59892 (−13)	2.66758 44662 51279 (−14)	6.70373 01203 98254 (−13)	−7.00552 44636 66701 (−14)
12	−2.84706 00495 41412 (−15)		7.31980 66017 91276 (−15)	

TABLE 3

Coefficients for the Series

$$x^{-\nu}J_\nu(x) = \sum_{n=0}^{\infty} A_n^{(\nu)}T_{2n}(x/5) \qquad x^{-\nu}I_\nu(x) = \sum_{n=0}^{\infty} B_n^{(\nu)}T_{2n}(x/5)$$

$$-5 \leqq x \leqq 5$$

n	$A_n^{(1/3)}$	$B_n^{(1/3)}$	$A_n^{(-1/3)}$	$B_n^{(-1/3)}$
0	8.94076 67920 14735 (−02)	6.58297 78114 19436 (+00)	−1.07166 32561 37673 (−01)	1.73346 79232 80504 (+01)
1	−4.81921 28594 44027 (−01)	7.20005 50672 62920 (+00)	−2.97571 62513 05484 (−01)	2.16272 09227 74341 (+01)
2	2.59994 46413 53814 (−01)	1.70885 10971 09031 (+00)	5.54730 81817 97741 (−01)	6.04768 89225 73269 (+00)
3	−5.17388 91497 92902 (−02)	2.19489 00443 12230 (−01)	−1.62312 94316 81218 (−01)	9.01519 59809 63211 (−01)
4	5.39598 09479 79882 (−03)	1.74386 20597 80771 (−02)	2.12830 59224 11643 (−02)	8.14844 58153 55754 (−02)
5	−3.48602 79717 45306 (−04)	9.35725 57411 65672 (−04)	−1.62033 34074 85423 (−03)	4.88835 76577 22940 (−03)
6	1.53744 49741 62028 (−05)	3.60628 89498 98151 (−05)	8.12833 49697 02066 (−05)	2.07687 29885 63908 (−04)
7	−4.93335 30371 51078 (−07)	1.04455 75512 24314 (−06)	−2.90085 92814 81608 (−06)	6.55696 66640 79319 (−06)
8	1.20489 92956 96342 (−08)	2.35404 20416 24506 (−08)	7.75818 87529 98852 (−08)	1.59597 72584 67654 (−07)
9	−2.31664 13965 03490 (−10)	4.24192 23858 96914 (−10)	−1.61479 04639 34516 (−09)	3.08285 63507 75112 (−09)
10	3.59939 64540 78735 (−12)	6.24808 74659 21528 (−12)	2.69219 41698 10025 (−11)	4.83729 62983 39093 (−11)
11	−4.61479 50001 45118 (−14)	7.66029 28778 25032 (−14)	−3.67795 65706 02520 (−13)	6.28465 97844 22833 (−13)
12			4.19393 32335 77453 (−15)	6.86947 00674 93498 (−15)

n	$A_n^{(2/3)}$	$B_n^{(2/3)}$	$A_n^{(-2/3)}$	$B_n^{(-2/3)}$
0	1.26821 19075 59155 (−01)	3.89700 09178 94441 (+00)	−1.34833 35738 84306 (−01)	2.70904 14407 24111 (+01)
1	−3.82426 30216 91411 (−01)	3.97522 01852 22248 (+00)	2.53684 77592 21459 (−01)	3.58034 51642 90575 (+01)
2	1.58419 46147 00555 (−01)	8.73294 88291 09913 (−01)	6.72159 99132 26151 (−01)	1.09071 34473 17453 (+01)
3	−2.73985 78534 08571 (−02)	1.04699 27477 53559 (−01)	−2.65206 75332 02099 (−01)	1.76188 79963 65858 (+00)
4	2.59903 60286 58541 (−03)	7.83606 87385 22999 (−03)	4.01486 63047 46445 (−02)	1.70744 74910 71144 (−01)
5	−1.56262 72721 66761 (−04)	3.99168 33880 23007 (−04)	−3.36287 90613 98914 (−03)	1.08765 98567 69296 (−02)
6	6.50257 66818 64264 (−06)	1.46964 86166 29454 (−05)	1.81307 01139 10497 (−04)	4.86813 02946 11161 (−04)
7	−1.98695 79839 64146 (−07)	4.08724 99803 03382 (−07)	−6.85821 46199 38979 (−06)	1.60889 95124 35949 (−05)
8	4.65189 13454 60930 (−09)	8.88083 87568 65976 (−09)	1.92618 60920 91631 (−07)	4.07864 59335 37985 (−07)
9	−8.61663 19643 98472 (−11)	1.54819 86261 88014 (−10)	−4.18242 83196 21301 (−09)	8.17158 76141 59425 (−09)
10	1.29480 93295 22571 (−12)	2.21246 66336 55002 (−12)	7.23806 22406 35861 (−11)	1.32536 13614 40354 (−10)
11	−1.61062 10063 91952 (−14)	2.63810 71950 41641 (−14)	−1.02241 47112 37617 (−12)	1.77479 93677 21115 (−12)
12			1.20165 09941 09705 (−14)	1.99468 46072 70085 (−14)

TABLE 3 (*Continued*)

n	$A_n^{(1/4)}$	$B_n^{(1/4)}$	$A_n^{(-1/4)}$	$B_n^{(-1/4)}$
0	7.21592 24626 52688 (−02)	7.47302 10303 05303 (+00)	−8.17099 31298 82310 (−02)	1.54456 46447 18517 (+01)
1	−4.97105 11867 57523 (−01)	8.31530 14410 19353 (+00)	−3.74456 45549 83537 (−01)	1.89744 72346 22373 (+00)
2	2.91288 83390 42546 (−01)	2.01309 13436 22401 (+00)	5.16008 82005 50960 (−01)	5.19510 68992 00228 (+00)
3	−6.02831 47437 67723 (−02)	2.63206 92238 80861 (−01)	−1.42226 29467 31262 (−01)	7.59501 67795 49208 (−01)
4	6.44904 67659 23062 (−03)	2.12367 20333 36512 (−02)	1.80565 04668 12229 (−02)	6.75005 48548 48750 (−02)
5	−4.24592 56017 28019 (−04)	1.15486 74718 87051 (−03)	−1.34437 14794 53047 (−03)	3.99099 14294 53922 (−03)
6	1.90112 60876 55165 (−05)	4.50337 94565 14874 (−05)	6.62934 97557 47040 (−05)	1.67428 98832 97653 (−04)
7	−6.17793 29918 73178 (−07)	1.31804 25685 19379 (−06)	−2.33304 02865 38664 (−06)	5.22735 04185 32855 (−06)
8	1.52537 19747 09771 (−08)	2.99821 23052 29824 (−08)	6.16613 44735 40195 (−08)	1.25977 09051 28136 (−07)
9	−2.96097 83840 28929 (−10)	5.44849 42083 22499 (−10)	−1.27028 56543 85534 (−09)	2.41177 00010 18902 (−09)
10	4.63993 80496 72974 (−12)	8.08732 15913 15397 (−12)	2.09864 07533 16501 (−11)	3.75370 60782 78502 (−11)
11	−5.99493 04600 18949 (−14)	9.98561 69608 87138 (−14)	−2.84373 93655 71641 (−13)	4.84076 50499 38746 (−13)
12		1.04140 77287 05166 (−15)	3.21872 59210 83734 (−15)	5.25514 33653 87754 (−15)

n	$A_n^{(3/4)}$	$B_n^{(3/4)}$	$A_n^{(-3/4)}$	$B_n^{(-3/4)}$
0	1.29259 76634 12324 (−01)	3.40367 50320 63991 (+00)	−1.07108 00067 60494 (−01)	3.01807 22091 21124 (+01)
1	−3.53485 41224 59797 (−01)	3.41159 09039 09293 (+00)	4.59282 45203 50027 (−01)	4.04154 70990 58814 (+01)
2	1.38693 46692 29917 (−01)	7.35518 91333 63896 (−01)	6.82534 33005 94923 (−01)	1.25813 97685 36136 (+01)
3	−2.32397 67410 71826 (−02)	8.67224 88421 14764 (−02)	−2.96708 11160 24287 (−01)	2.07484 85659 26079 (+00)
4	2.15622 40367 90532 (−03)	6.39730 24996 65543 (−03)	4.67644 95998 05034 (−02)	2.04728 16790 32805 (−01)
5	−1.27441 43552 17310 (−04)	3.21786 27908 36382 (−04)	−4.01877 21906 54069 (−03)	1.32448 04086 74998 (−02)
6	5.22966 96656 56027 (−06)	1.17162 12539 26833 (−05)	2.20820 19108 00076 (−04)	6.00806 99443 67353 (−04)
7	−1.57920 21138 77838 (−07)	3.22621 49048 88015 (−07)	−8.48036 03378 59174 (−06)	2.00909 87697 36448 (−05)
8	3.65942 38094 19545 (−09)	6.94760 17547 32315 (−09)	2.41209 84427 69199 (−07)	5.14647 54652 28697 (−07)
9	−6.71688 65154 82103 (−11)	1.20138 77599 87366 (−10)	−5.29481 93781 41679 (−09)	1.04076 51295 38435 (−08)
10	1.00112 47892 05989 (−12)	1.70415 17978 78482 (−12)	9.25118 40343 21647 (−11)	1.70234 23895 70753 (−10)
11	−1.23610 48693 00485 (−14)	2.01814 93824 57457 (−14)	−1.31797 96261 75435 (−12)	2.29723 86885 18947 (−12)
12			1.56102 37245 93056 (−14)	2.60017 90694 76929 (−14)

$N = [n(n + 2\alpha + 1)]^{1/2}$, $\gamma = -(1 + 2\alpha)/4$. In general, if values of $f(x)$ for complex x are desired, it is wisest to choose a such that the expansions are interpolatory along a suitable ray in the complex x-plane and to stay as close as possible to this ray.

Suppose we have the truncated expansion

$$f(x) = \sum_{n=0}^{N} A_n T_n(x) + \epsilon_{N+1} = \phi_N(x) + \epsilon_{N+1}, \quad -1 \leqq x \leqq 1, \tag{5.2}$$

and

$$\epsilon_{N+1} = \sum_{n=N+1}^{\infty} A_n T_n(x) \tag{5.3}$$

Then $\phi_N(x)$ is not generally the Chebyshev approximation of degree N to $f(x)$ in the sense of [11], i.e., the polynomial $\Phi_N(x)$ of degree N uniquely characterized by the fact that in the interval $[-1, 1]$ the number of consecutive points at which the difference $f(x) - \Phi_N(x)$ with alternate changes in sign assumes the value

$$\max_{-1 \leqq x \leqq 1} |f(x) - \Phi_N(x)|,$$

is not less than $N + 2$; but $\phi_N(x)$ may closely approximate $\Phi_N(x)$. How closely, of course, depends on the coefficients A_n. If A_n goes quite rapidly to zero as $n \to \infty$, then A_{N+2} is small compared to A_{N+1} and consequently

$$\epsilon_{N+1} \sim A_{N+1} T_{N+1}(x) \tag{5.4}$$

and the error curve is practically uniform, i.e., $\phi_N(x)$ is nearly $\Phi_N(x)$. Such is the case in our expansions, and, consequently, we must expect the approximation $\Phi_N(x)$ for moderate values of a to offer a negligible improvement over the Chebyshev polynomial expansions derived in this paper and truncated after $N + 1$ terms.

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Expansions of Hypergeometric Functions in Hypergeometric Functions

By Jerry L. Fields and Jet Wimp

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Abstract. In [1] Luke gave an expansion of the confluent hypergeometric function in terms of the modified Bessel functions $I_\nu(z)$. The existence of other, similar expansions implied that more general expansions might exist. Such was the case. Here multiplication type expansions of low-order hypergeometric functions in terms of other hypergeometric functions are generalized by Laplace transform techniques.

1. General Expansions. The generalized hypergeometric function ${}_pF_q(z)$, [2], is defined by

$$(1.1)\quad \begin{cases} {}_pF_q(z) = {}_pF_q\left(\begin{matrix} a_1, \cdots, a_p \\ b_1, \cdots, b_q \end{matrix} \middle| z\right) = \sum_{k=0}^{\infty} \dfrac{\prod_{j=1}^{p} (a_j)_k}{\prod_{j=1}^{q} (b_j)_k} \cdot \dfrac{z^k}{k!}, \\ \text{where} \qquad (\sigma)_\mu = \dfrac{\Gamma(\sigma+\mu)}{\Gamma(\sigma)}. \end{cases}$$

We assume that no a_j is equal to any b_j and that no b_j is a negative integer. For ease in writing, we employ the contracted notation

$$(1.2)\qquad {}_pF_q(z) = {}_pF_q\left(\begin{matrix} a_p \\ b_q \end{matrix} \middle| z\right) = \sum_{k=0}^{\infty} \frac{(a_p)_k}{(b_q)_k} \cdot \frac{z^k}{k!}.$$

Thus $(a_p)_k$ is to be interpreted as $\prod_{j=1}^{p} (a_j)_k$ and similarly for $(b_q)_k$. Considered as a power series in z, ${}_pF_q(z)$ has a radius of convergence equal to infinity if $p \leqq q$ and equal to unity if $p = q + 1$. In general, ${}_pF_q(z)$ is not defined if $p \geqq q + 2$. However, in this paper, we shall say that ${}_pF_q(zw)$ is equal to another series if the coefficients of $(w)^k$ on both sides are equal, regardless of the relationship between p and q.

Our first expansion is

$$(1.3)\qquad \begin{aligned} {}_{p+r}F_{q+s}\left(\begin{matrix} a_p, c_r \\ b_q, d_s \end{matrix} \middle| zw\right) &= \sum_{n=0}^{\infty} \frac{(a_p)_n(\alpha)_n(\beta)_n(-z)^n}{(b_q)_n(\gamma+n)_n\, n!} \\ &\times {}_{p+2}F_{q+1}\left(\begin{matrix} n+\alpha, n+\beta, n+a_p \\ 2n+\gamma+1, n+b_q \end{matrix} \middle| z\right) {}_{r+2}F_{s+2}\left(\begin{matrix} -n, n+\gamma, c_r \\ \alpha, \beta, d_s \end{matrix} \middle| w\right). \end{aligned}$$

From (1.3) we may derive other expansions by confluence:

$$(1.4)\qquad \begin{aligned} {}_{p+r}F_{q+s}\left(\begin{matrix} a_p, c_r \\ b_q, d_s \end{matrix} \middle| zw\right) &= \sum_{n=0}^{\infty} \frac{(a_p)_n(\alpha)_n(-z)^n}{(b_q)_n(\gamma+n)_n\, n!} \\ &\times {}_{p+1}F_{q+1}\left(\begin{matrix} n+\alpha, n+a_p \\ 2n+\gamma+1, n+b_q \end{matrix} \middle| z\right) {}_{r+2}F_{s+1}\left(\begin{matrix} -n, n+\gamma, c_r \\ \alpha, d_s \end{matrix} \middle| w\right), \end{aligned}$$

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$$(1.5)\quad {}_{p+r}F_{q+s}\left(\begin{matrix}a_p, c_r\\ b_q, d_s\end{matrix}\middle| zw\right) = \sum_{n=0}^{\infty} \frac{(a_p)_n(\alpha)_n(-z)^n}{(b_q)_n n!}\, {}_{p+1}F_q\left(\begin{matrix}n+\alpha, n+a_p\\ n+b_q\end{matrix}\middle| z\right) \times {}_{r+1}F_{s+1}\left(\begin{matrix}-n, c_r\\ \alpha, d_s\end{matrix}\middle| w\right),$$

$$(1.6)\quad {}_{p+r}F_{q+s}\left(\begin{matrix}a_p, c_r\\ b_q, d_s\end{matrix}\middle| zw\right) = \sum_{n=0}^{\infty} \frac{(a_p)_n(-z)^n}{(b_q)_n n!}\, {}_pF_q\left(\begin{matrix}n+a_p\\ n+b_q\end{matrix}\middle| z\right) \times {}_{r+1}F_s\left(\begin{matrix}-n, c_r\\ d_s\end{matrix}\middle| w\right).$$

For example, if in (1.3) we replace r by $r+1$, z by z/c_{r+1}, set $\beta = c_{r+1}$ and let $\beta \to \infty$, (1.4) results; (1.5) and (1.6) may be similarly derived; (1.6) is a result given previously by Meijer, [7], 1953, page 355, but it is the only result above which can be deduced from his work.

Equation (1.3) is proved by induction on p, q, r, and s. The case $p = q = r = s = 0$ reduces to

$$(1.7)\quad e^{zw} = \sum_{n=0}^{\infty} \frac{(\alpha)_n(\beta)_n(-z)^n}{(\gamma+n)_n n!}\, {}_2F_1\left(\begin{matrix}n+\alpha, n+\beta\\ 2n+\gamma+1\end{matrix}\middle| z\right) {}_2F_2\left(\begin{matrix}-n, n+\gamma\\ \alpha, \beta\end{matrix}\middle| w\right),$$

a result given by Luke, see (1.8) of [3]. The proof of (1.3) then proceeds by Laplace and inverse-Laplace transform techniques. Assume (1.3) true for p, q, r and s, multiply both sides of (1.3) by $(w)^{\sigma-1}$, take the Laplace transform of both sides with respect to w, and we obtain, see (17), page 219 of [4],

$$\int_0^{\infty} e^{-\lambda w} w^{\sigma-1}\, {}_{p+r}F_{q+s}\left(\begin{matrix}a_p, c_r\\ b_q, d_s\end{matrix}\middle| zw\right) dw = \frac{\Gamma(\sigma)}{(\lambda)^{\sigma}}\, {}_{p+r+1}F_{q+s}\left(\begin{matrix}a_p, c_r, \sigma\\ b_q, d_s\end{matrix}\middle| \frac{z}{\lambda}\right)$$

$$(1.8)\quad = \frac{\Gamma(\sigma)}{(\lambda)^{\sigma}} \sum_{n=0}^{\infty} \frac{(a_p)_n(\alpha)_n(\beta)_n(-z)^n}{(b_q)_n(\gamma+n)_n n!}\, {}_{p+2}F_{q+1}\left(\begin{matrix}n+\alpha, n+\beta, n+a_p\\ 2n+\gamma+1, n+b_q\end{matrix}\middle| z\right) \times {}_{r+3}F_{s+2}\left(\begin{matrix}-n, n+\gamma, c_r, \sigma\\ \alpha, \beta, d_s\end{matrix}\middle| \frac{1}{\lambda}\right).$$

The induction on r is completed by replacing $1/\lambda$ by w in (1.10). The induction with respect to s is effected by multiplying both sides of (1.3) by $(w)^{\sigma}$, letting $w = 1/\lambda$ and applying the inverse-Laplace transform, see (1), page 297 of [4]. If the above Laplace transform techniques are applied to z instead of w, the inductions on p and q can be similarly effected.

2. Specialized Expansions. In this section we give several interesting cases of (1.3)–(1.6).

From (1.3) we have

$$(2.1)\quad {}_rF_s\left(\begin{matrix}c_r\\ d_s\end{matrix}\middle| zw\right) = \frac{2^{\gamma}}{[1+(1-z)^{1/2}]^{\gamma}} \sum_{n=0}^{\infty} \frac{(\gamma)_n(-z)^n}{n![1+(1-z)^{1/2}]^{2n}} \times {}_{r+2}F_{s+2}\left(\begin{matrix}-n, n+\gamma, c_r\\ \gamma/2, (\gamma+1)/2, d_s\end{matrix}\middle| w\right),$$

where we have used the relation, see [2], page 101, (6),

$$(2.2)\qquad \left[\frac{1}{2}+(1-z)^{1/2}/2\right]^{1-2a} = {}_2F_1\left(\begin{matrix}a-\frac{1}{2}, a\\ 2a\end{matrix}\middle| z\right).$$

The expansion

$$(2.3)\qquad \begin{aligned} {}_rF_s\left(\begin{matrix}c_r\\ d_s\end{matrix}\middle| zw\right) = \frac{e^{z/2}}{(\frac{1}{2})_\gamma(z)^\gamma}\Bigg\{&2^{2\gamma}(1)_\gamma(\tfrac{1}{2})_\gamma I_\gamma(z/2)\\ &+2\sum_{n=1}^\infty(-1)^n\frac{(n+\gamma)}{n}(n)_{2\gamma}\,{}_{r+2}F_{s+1}\left(\begin{matrix}-n, n+2\gamma, c_r\\ \frac{1}{2}+\gamma, d_s\end{matrix}\middle| w\right) I_{n+\gamma}\left(\frac{z}{2}\right)\Bigg\}\end{aligned}$$

follows from (1.4), where $I_\nu(z)$ is the modified Bessel function of the first kind. Here, use is made of the expression

$$(2.4)\qquad \Gamma(\nu+1)\, I_\nu(x) = (x/2)^\nu e^{-x}\, {}_1F_1\left(\begin{matrix}\frac{1}{2}+\nu\\ 1+2\nu\end{matrix}\middle| 2x\right).$$

Also from (1.4) we get

$$(2.5)\qquad \begin{aligned} {}_pF_q\left(\begin{matrix}a_p\\ b_q\end{matrix}\middle| zw\right) &= {}_{p+1}F_{q+1}\left(\begin{matrix}\beta+1, a_p\\ \alpha+\beta+2, b_q\end{matrix}\middle| z\right)\\ &+\sum_{n=1}^\infty \frac{(a_p)_n(z)^n}{(b_q)_n(n+\alpha+\beta+1)_n}\,{}_{p+1}F_{q+1}\left(\begin{matrix}n+\beta+1, n+a_p\\ 2n+\alpha+\beta+2, n+b_q\end{matrix}\middle| z\right)\\ &\qquad\times P_n^{(\alpha,\beta)}(2w-1),\end{aligned}$$

where

$$(2.6)\qquad P_n^{(\alpha,\beta)}(x) = (-1)^n\frac{(1+\beta)_n}{n!}\,{}_2F_1\left(\begin{matrix}-n, n+\alpha+\beta+1\\ 1+\beta\end{matrix}\middle| \frac{1+x}{2}\right).$$

The $P_n^{(\alpha,\beta)}(x)$ are known as the Jacobi polynomials, see [8], and reduce to the Chebyshev polynomials of the first kind if $\alpha=\beta=-\frac{1}{2}$.

Equation (1.5) yields the following expansions:

$$(2.7)\qquad {}_rF_{s+1}\left(\begin{matrix}c_r\\ d_s, \delta\end{matrix}\middle| zw\right) = e^z\sum_{n=0}^\infty\frac{(-z)^n}{n!}\,{}_{r+1}F_{s+1}\left(\begin{matrix}-n, c_r\\ \delta, d_s\end{matrix}\middle| w\right);$$

$$(2.8)\qquad {}_rF_s\left(\begin{matrix}c_r\\ d_s\end{matrix}\middle| zw\right) = (1-z)^{-\alpha}\sum_{n=0}^\infty\frac{(\alpha)_n}{n!}\left(\frac{z}{z-1}\right)^n {}_{r+1}F_{s+1}\left(\begin{matrix}-n, c_r\\ \alpha, d_s\end{matrix}\middle| w\right);\quad\text{and}$$

$$(2.9)\qquad {}_rF_{s+1}\left(\begin{matrix}c_r\\ d_s, 1+\delta\end{matrix}\middle| -zw\right) = \frac{\delta}{(z)^\delta}\sum_{n=0}^\infty\frac{\gamma(\delta+n, z)}{n!}\,{}_{r+1}F_{s+1}\left(\begin{matrix}-n, c_r\\ \delta, d_s\end{matrix}\middle| w\right),$$

where

$$(2.10)\qquad \begin{cases}\gamma(a,z) = \Gamma(a) - \displaystyle\int_z^\infty e^{-t}t^{a-1}\,dt,\\[2mm] \qquad\quad = a^{-1}z^a\,{}_1F_1\left(\begin{matrix}a\\ a+1\end{matrix}\middle| -z\right).\end{cases}$$

$$(2.11)\qquad {}_pF_q\left(\begin{matrix}a_p\\ b_q\end{matrix}\middle| zw\right) = \sum_{n=0}^\infty\frac{(a_p)_n(-z)^n}{(b_q)_n}\,{}_{p+1}F_q\left(\begin{matrix}n+\alpha+1, n+a_p\\ n+b_q\end{matrix}\middle| z\right) L_n^\alpha(w),$$

where

$$L_n^{\alpha}(x) = \frac{(\alpha+1)_n}{n!}\,{}_1F_1\left(\begin{matrix}-n\\1+\alpha\end{matrix}\middle|\,x\right). \tag{2.12}$$

The $L_n^{\alpha}(x)$ are known as the generalized Laguerre polynomials, see [8].

The result (2.7) can also be deduced from the work of Rainville [5], page 267, (25), and (2.8) is a generalization of a result given by Chaundy [6] and Meijer [7], 1952, page 483.

If we use the formula

$${}_0F_1\left(\nu+1\,\middle|\,\frac{x^2}{4}\right) = \Gamma(\nu+1)\left(\frac{2}{x}\right)^{\nu} I_\nu(x), \tag{2.13}$$

(1.6) yields the expansion

$$\begin{aligned}&{}_rF_{s+1}\left(\begin{matrix}c_r\\d_s,\,1+\delta\end{matrix}\middle|\,\frac{z^2w}{4}\right)\\ &\qquad= \Gamma(1+\delta)\left(\frac{2}{z}\right)^{\delta}\sum_{n=0}^{\infty}\frac{\left(\frac{-z}{2}\right)^n}{n!}\,I_{\delta+n}(z)\,{}_{r+1}F_s\left(\begin{matrix}-n,\,c_r\\d_s\end{matrix}\middle|\,w\right).\end{aligned} \tag{2.14}$$

Since the merits of modified Bessel function expansions are well known, we consider the convergence properties of (2.3) in more detail. They are determined principally by the asymptotic properties of the polynomials

$${}_{r+2}F_{s+1}\left(\begin{matrix}-n,\,n+2\gamma,\,c_r\\\frac{1}{2}+\gamma,\,d_s\end{matrix}\middle|\,w\right)$$

for large n. For an extensive treatment of the asymptotic properties of these polynomials for large n, see [9]. In general, the convergence properties of (2.3) are superior to the original series definition if the polynomials

$${}_{r+2}F_{s+1}\left(\begin{matrix}-n,\,n+2\gamma,\,c_r\\\frac{1}{2}+\gamma,\,d_s\end{matrix}\middle|\,w\right)$$

are interpolatory or nearly so. For example, if $r = s$, $w = 1$ and $c_i - c_j$ is never an integer or zero, for $i \neq j$,

$$\begin{aligned}&{}_{r+2}F_{r+1}\left(\begin{matrix}-n,\,n+2\gamma,\,c_r\\\frac{1}{2}+\gamma,\,d_r\end{matrix}\middle|\,1\right) = \left[An^{2\mu} + \sum_{t=1}^{r}B_t n^{-2c_t}\right]\cdot\left[1+0\left(\frac{1}{n}\right)\right],\\ &\qquad\mu = \sum_{t=1}^{r}c_t - \sum_{t=1}^{r}d_t,\end{aligned} \tag{2.15}$$

and the A and B_t's are constants independent of n. Incorporating the inequality, see (1), page 49 of [10],

$$|I_\nu(z)| \leq \frac{|z/2|^{\nu}}{\Gamma(\nu+1)}\exp\{Im(z)\}, \tag{2.16}$$

we see that (2.3) converges like

$$\sum_{n=1}^{\infty}\left\{\frac{(n)^{2\mu+\gamma}}{n!4^n} + \sum_{t=1}^{r}\frac{n^{\gamma-2c_t}}{n!4^n}\right\}|z|^n. \tag{2.17}$$

The original series definition, however, converges like

$$\sum_{n=1-n}^{\infty} \frac{n^{\mu}}{n!} |z|^{n}. \tag{2.18}$$

3. Further Expansions Involving Free Parameters. If in (1.3), we replace p by $p - 2$ and set $\alpha = a_{p-1}$, $\beta = a_p$, $w = 0$, we get

$$1 = \sum_{n=0}^{\infty} \frac{(a_p)_n(-z)^n}{(b_q)_n(\gamma + n)_n n!} {}_pF_{q+1}\left(\begin{matrix} n + a_p \\ n + b_q, 2n + \gamma + 1 \end{matrix}\middle| z\right). \tag{3.1}$$

Then replacing a_p by $a_p + k$, b_q by $b_q + k$, γ by $\gamma + 2k$ and multiplying both sides of (3.1) by

$$\frac{(c_r)_k(zw)^k}{(d_s)_k k!},$$

we get

$$\begin{aligned} \frac{(c_r)_k(zw)^k}{(d_s)_k k!} &= \frac{(c_r)_k(zw)^k}{(d_s)_k k!} \sum_{n=0}^{\infty} \frac{(k + a_p)_n(-z)^n}{(k + b_q)_n(n + 2k + \gamma)_n n!} \\ &\qquad \times {}_pF_{q+1}\left(\begin{matrix} n + k + a_p \\ n + k + b_q, 2n + 2k + \gamma + 1 \end{matrix}\middle| z\right) \\ &= \sum_{n=k}^{\infty} \frac{(a_p)_n(-z)^n}{(b_q)_n(\gamma + n)_n n!} {}_pF_{q+1}\left(\begin{matrix} n + a_p \\ n + b_q, 2n + \gamma + 1 \end{matrix}\middle| z\right) \\ &\qquad \times \frac{(-n)_k(n + \gamma)_k(b_q)_k(c_r)_k(w)^k}{(a_p)_k(d_s)_k k!}. \end{aligned} \tag{3.2}$$

Summing the terms represented in (3.2) from k equal zero to infinity, and interchanging the summation processes with respect to k and n, we obtain

$$\begin{aligned} {}_rF_s\left(\begin{matrix} c_r \\ d_s \end{matrix}\middle| zw\right) &= \sum_{n=0}^{\infty} \frac{(a_p)_n(-z)^n}{(b_q)_n(\gamma + n)_n n!} {}_pF_{q+1}\left(\begin{matrix} n + a_p \\ n + b_q, 2n + \gamma + 1 \end{matrix}\middle| z\right) \\ &\qquad \times {}_{q+r+2}F_{p+s}\left(\begin{matrix} -n, n + \gamma, b_q, c_r \\ a_p, d_s \end{matrix}\middle| w\right). \end{aligned} \tag{3.3}$$

Using the Laplace transform techniques of Section 1, we arrive at the expansion

$$\begin{aligned} {}_{r+t}F_{s+u}\left(\begin{matrix} c_r, e_t \\ d_s, f_u \end{matrix}\middle| zw\right) &= \sum_{n=0}^{\infty} \frac{(e_t)_n(a_p)_n(-z)^n}{(f_u)_n(b_q)_n(\gamma + n)_n n!} \\ &\qquad \times {}_{p+t}F_{q+u+1}\left(\begin{matrix} n + a_p, n + e_t \\ n + b_q, n + f_u, 2n + \gamma + 1 \end{matrix}\middle| z\right) \\ &\qquad \times {}_{q+r+2}F_{p+s}\left(\begin{matrix} -n, n + \gamma, b_q, c_r \\ a_p, ds \end{matrix}\middle| w\right). \end{aligned} \tag{3.4}$$

Equation (3.4) and its confluent form not containing γ are generalizations of results given by Chaundy, see (11), page 187 of [2].

As a final example of how Laplace transform techniques may be used in general expansions, we prove

$$(3.5)\quad {}_pF_q\left(\begin{matrix} a_p \\ b_q \end{matrix}\middle| \lambda z\right) {}_rF_s\left(\begin{matrix} c_r \\ d_s \end{matrix}\middle| \mu z\right) = \sum_{n=0}^{\infty} \frac{(a_p)_n(\lambda z)^n}{(b_q)_n n!} \times {}_{r+q+1}F_{p+s}\left(\begin{matrix} -n, 1-n-b_q, c_r \\ 1-n-a_p, d_s \end{matrix}\middle| \frac{(-1)^{p+q+1}\mu}{\lambda}\right).$$

Again the proof proceeds by induction on p, q, r and s. The case $p = q = r = s = 0$ reduces to

$$(3.6)\quad e^{\lambda z}\cdot e^{\mu z} = \sum_{n=0}^{\infty} \frac{(\lambda+\mu)^n z^n}{n!} = \sum_{n=0}^{\infty} \frac{(\lambda z)^n}{n!} {}_1F_0\left(-n\middle| -\frac{\mu}{\lambda}\right),$$

since

$$(3.7)\quad {}_1F_0(\alpha \mid z) = (1-z)^{-\alpha}.$$

Assuming (3.5) true for p, q, r and s, the inductions with respect to r and s can be effected by using with respect to μ the Laplace transform techniques illustrated in the proof of (1.3). The inductions with respect to p and q are similarly carried out with respect to λ after making use of the relationship

$$(3.8)\quad {}_{1+p}F_q\left(\begin{matrix} -n, a_p \\ b_q \end{matrix}\middle| z\right) = \frac{(a_p)_n(-z)^n}{(b_q)_n} {}_{1+q}F_p\left(\begin{matrix} -n, 1-n-b_q \\ 1-n-a_p \end{matrix}\middle| \frac{(-1)^{q+p}}{z}\right).$$

This completes the induction proof. Equation (3.5) is a generalization of results given by Chaundy, see (12)–(15), page 187 [2]*.

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* There are two misprints in these formulas. In (12), p/q should be replaced by q/p and in (13), $-a'$ by a' on the right side of the equation.

Polynomial Approximations to Integral Transforms

By Jet Wimp

1. Introduction. The symmetric Jacobi polynomials $P_n^{(\alpha,\alpha)}(x)$, orthogonal on the interval $-1 \leqq x \leqq 1$, are widely used for approximating functions, but the integral which defines the coefficients for the expansion of a function $g(x)$ in these polynomials usually is quite difficult to evaluate. The problem is simplified if $g(x)$ is an integral transform of the Fourier or Laplace type, since the kernel of the transform generates a series of the above polynomials. The coefficients in such cases are found to be Hankel transforms, which are widely tabulated.

Examples include Chebyshev polynomial expansions of $1/(x + a)^k$, $\psi(x + a)$, $\log \Gamma(x + a)$, $Ci(x)$ and $Si(x)$.

2. Formulas When $g(x)$ is a Laplace or Fourier Transform. The symmetric Jacobi polynomials [1, v. 2, p. 168] may be defined by

$$P_n^{(\alpha,\alpha)}(x) = \binom{n+\alpha}{n} {}_2F_1[-n, n + 2\alpha + 1; \alpha + 1; \tfrac{1}{2} - \tfrac{1}{2}x]. \tag{1}$$

A function $g(x)$ satisfying certain conditions has the expansion

$$g(x) = \sum_{n=0}^{\infty} A_n P_n^{(\alpha,\alpha)}(x), \qquad -1 \leqq x \leqq 1, \tag{2}$$

where

$$A_n = \frac{(2n + 2\alpha + 1)n!\Gamma(n + 2\alpha + 1)}{2^{2\alpha+1}[\Gamma(n + \alpha + 1)]^2} \int_{-1}^{1} g(x)(1 - x^2)^\alpha P_n^{(\alpha,\alpha)}(x)\, dx. \tag{3}$$

Suppose now that $g(x)$ is the Laplace transform of some $f(t)$,

$$g(x) = \mathfrak{L}\{f(t)\} = \int_0^\infty e^{-xt} f(t)\, dt = \sum_{n=0}^{\infty} A_n P_n^{(\alpha,\alpha)}(x). \tag{4}$$

To determine the A_n's replace the kernel of the Laplace transform by its Neumann series [1, v. 2: p. 98, No. (1); p. 175, No. (16); p. 174, No. (6); and the duplication formula for the gamma function].

$$e^{-xt} = \sum_{n=0}^{\infty} (-)^n \Omega_n \frac{I_{n+\alpha+1/2}(t)}{t^{\alpha+1/2}} P_n^{(\alpha,\alpha)}(x), \tag{5}$$

$$\Omega_n = \frac{2^{1/2-\alpha}\pi^{1/2}(n + \alpha + \frac{1}{2})\Gamma(n + 2\alpha + 1)}{\Gamma(n + \alpha + 1)}. \tag{6}$$

Then (4) yields

$$A_n = e^{(n-\alpha-1)[\pi i/2]}\Omega_n \mathfrak{H}\left\{\frac{f(t)}{t^{\alpha+1}}\right\}_{\substack{y=i \\ \nu=n+\alpha+1/2}}, \tag{7}$$

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$$\mathcal{H}\{F(t)\} = \int_0^\infty F(t)J_\nu(yt)(yt)^{1/2}\,dt. \tag{8}$$

$\mathcal{H}\{F(t)\}$ denotes the Hankel transform of $F(t)$ [2].

When $\alpha = -\frac{1}{2}$, (7) furnishes the coefficients for the Chebyshev expansion

$$g(x) = \int_0^\infty e^{-xt}f(t)\,dt = \sum_{n=0}^\infty C_n\,T_n(x), \qquad -1 \leqq x \leqq 1, \tag{9}$$

where

$$C_n = \epsilon_n\,e^{(n-1/2)[\pi i/2]}\mathcal{H}\left\{\frac{f(t)}{t^{1/2}}\right\}_{\substack{y=i\\ \nu=n}}, \qquad \epsilon_n = \begin{matrix}1, n = 0,\\ 2, n > 0.\end{matrix} \tag{10}$$

If we replace t by it in (5), we find that the same method is applicable when $g(x)$ is a Fourier transform of $f(t)$. We omit details, but the key results for the sine and cosine transforms are as follows.

$$\begin{matrix} g_1(x)\\ g_2(x)\end{matrix} = \int_0^\infty f(t)\begin{matrix}\sin\\ \cos\end{matrix}(xt)\,dt = \sum_{n=0}^\infty \begin{matrix}S_n\\ C_n\end{matrix}\,P_n^{(\alpha,\alpha)}(x), \qquad -1 \leqq x \leqq 1 \tag{11}$$

where

$$S_n = \begin{cases}0, & n \text{ even},\\ e^{(n-1)[\pi i/2]}\Omega_n\mathcal{H}\left\{\dfrac{f(t)}{t^{\alpha+1}}\right\}_{\substack{y=1\\ \nu=n+\alpha+1/2}}, & n \text{ odd},\end{cases} \tag{12}$$

and

$$C_n = \begin{cases}0, & n \text{ odd},\\ e^{n\pi i/2}\Omega_n\,\mathcal{H}\left\{\dfrac{f(t)}{t^{\alpha+1}}\right\}_{\substack{y=1\\ \nu=n+\alpha+1/2}}, & n \text{ even}.\end{cases} \tag{13}$$

3. The Chebyshev Expansion for $1/(y + a)^k$. Let $g(x) = \left[\dfrac{x+1}{2} + a\right]^{-k}$. Then

$$\mathfrak{L}^{-1}\{g(x)\} = \frac{2^k}{(k-1)!}\,e^{-(2a+1)t}t^{k-1} = f(t). \tag{14}$$

Use (10) and let $y = \dfrac{x+1}{2}$. Then $T_n(2y-1) = T_n^*(y)$, $0 \leqq y \leqq 1$, is the shifted Chebyshev polynomial [3] and

$$\begin{aligned}\frac{1}{(y+a)^k} = &\left\{\sum_{n=0}^\infty \frac{\epsilon_n(-)^n(k+n-1)!}{(k-1)!}\,P_{k-1}^{-n}\right.\\ &\left.\cdot\left[\frac{2a+1}{2\sqrt{a^2+a}}\right]T_n^*(y)\right\}\Big/\,(a^2+a)^{k/2} \qquad 0 \leqq y \leqq 1, \quad a > 0,\end{aligned} \tag{15}$$

where $P_\nu^\mu(x)$ is the Legendre function [1, v. 1, p. 120]. For $k = 1$, (15) agrees with a result of Luke [4].

TABLE 1

Coefficients for the Series

$$\psi(x+a) = \sum_{n=0}^{\infty} C_n T_n(x), \qquad \ln \Gamma(x+a) = \sum_{n=0}^{\infty} S_n T_n(x).$$

n	$a = 2$		$a = 3$		$a = 4$		$a = 5$	
	C_n	S_n	C_n	S_n	C_n	S_n	C_n	S_n
0	0.30459199	0.17002422	0.88194225	0.79383494	1.23549564	1.86343494	1.49369453	3.23372482
1	.72037978	.36686678	.41097870	.90276517	.28965835	1.24591092	.22406724	1.49994422
2	−.12454959	.17315258	−.04164582	.10135581	−.02083054	.07191856	−.01249938	.05578533
3	.02776946	−.01962889	.00555546	−.00680364	.00198412	−.00343655	.00092592	−.00207042
4	−.00677624	.00325570	−.00082401	.00067831	−.00021127	.00024503	−.00007686	.00011489
5	.00172388	−.00063281	.00012898	−.00008032	.00002385	−.00002085	.00000678	−.00000762
6	−.00044818	.00013383	−.00002083	.00001046	−.00000279	.00000196	−.00000062	.00000056
7	.00011794	−.00002978	.00000343	−.00000145	.00000033	−.00000020	.00000006	−.00000004
8	−.00003125	.00000685	−.00000057	.00000021	−.00000004	.00000002	−.00000001	—
9	.00000832	−.00000161	.00000010	−.00000003	—	—	—	—
10	−.00000222	.00000039	−.00000002	—	—	—	—	—
11	.00000059	−.00000009	—	—	—	—	—	—
12	−.00000016	.00000002	—	—	—	—	—	—
13	.00000004	−.00000001	—	—	—	—	—	—
14	−.00000001	—	—	—	—	—	—	—
15	—	—	—	—	—	—	—	—

4. The Psi and Log Gamma Functions. These examples show how a property of the Laplace transform may be used to advantage when applying (4) and (8). We know that

$$\mathfrak{L}\{e^{-at}f(t)\} = g(x + a). \tag{16}$$

If $g(x)$ cannot be expanded in symmetric Jacobi polynomials, a in (16) can often be chosen so that $g(x + a)$ has a convergent expansion. Let

$$g(x) = \psi^{(m)}(x) = D^{m+1}\log\Gamma(x). \tag{17}$$

Since $\psi^{(m)}(x)$ has poles at zero and the negative integers, we cannot expand the function over $-1 \leqq x \leqq 1$. However, if

$$g(x) = \psi^{(m)}(x + a), \tag{18}$$

then

$$f(t) = \mathfrak{L}^{-1}\{g(x)\} = (-)^{m+1}e^{-at}t^m[1 - e^{-t}]^{-1}, \tag{19}$$

and if $\mathrm{Re}(a) > 1$, (7), and in particular (10), may be used since (18) is analytic for $|x| \leqq 1$. Substituting (19) in (10) and expanding $(1 - e^{-t})^{-1}$ by the binomial theorem, we have

$$C_n = -\epsilon_n \sum_{k=0}^{\infty} \frac{d^m}{dx^m}\left[\frac{(\sqrt{x^2 - 1} - x)^n}{\sqrt{x^2 - 1}}\right]\Bigg|_{x=k+a}. \tag{20}$$

Setting m equal to zero, we get

$$C_n = -\epsilon_n \sum_{k=0}^{\infty} \frac{[\sqrt{(k+a)^2 - 1} - (k + a)]^n}{\sqrt{(k + a)^2 - 1}}, \qquad n \geqq 1. \tag{21}$$

TABLE 2

Coefficients for the Series

$$Ci(x) = \int_{\infty}^{x} \frac{\cos t}{t}\,dt = \log(x) + \sum_{n=0}^{\infty} A_{2n}T_{2n}\left(\frac{x}{a}\right), \qquad 0 < x \leqq a$$

$$Si(x) = \int_{0}^{x} \frac{\sin t}{t}\,dt = \sum_{n=0}^{\infty} B_{2n+1}T_{2n+1}\left(\frac{x}{a}\right), \qquad -a \leqq x \leqq a$$

n	$a = 2$		$a = 5$	
	A_{2n}	B_{2n+1}	A_{2n}	B_{2n+1}
0	0.13529 62627	1.69809 09708	−0.96313 15550	2.08578 21107
1	−.42327 51922	−.09558 49521	−1.13103 16550	−.67042 59749
2	.01822 27219	.00295 78196	.34661 70891	.15186 68742
3	−.00041 57650	−.00005 14215	−.05698 43620	−.01861 43512
4	.00000 56716	.00000 05642	.00537 47844	.00138 96747
5	−.00000 00511	−.00000 00042	−.00032 52237	−.00006 95137
6	.00000 00003	—	.00001 36729	.00000 24908
7	—	—	−.00000 04226	−.00000 00671
8	—	—	.00000 00100	.00000 00014
9	—	—	−.00000 00002	—

If $n = 0$, (21) diverges, and for $n = 1$ the series is slowly convergent, but since $T_n(1) = 1$, $T_n(-1) = (-)^n$, we may solve for C_0 and C_1 in terms of higher computable coefficients, i.e.,

$$\text{(22)} \qquad \begin{cases} C_0 = \dfrac{\psi(a+1) + \psi(a-1)}{2} - \displaystyle\sum_{k=1}^{\infty} C_{2k}, \\ C_1 = \dfrac{\psi(a+1) - \psi(a-1)}{2} - \displaystyle\sum_{k=1}^{\infty} C_{2k+1}. \end{cases}$$

Integration of the series defined by (21) yields a Chebyshev expansion for $\ln \Gamma(x + a)$ because [3]

$$\text{(23)} \qquad \int T_n(x)\, dx = \frac{1}{2}\left[\frac{T_{n+1}(x)}{n+1} - \frac{T_{n-1}(x)}{n-1}\right] + C.$$

In Table 1 are listed coefficients for the Chebyshev expansions of $\psi(x + a)$ and $\log \Gamma(x + a)$, $a = 2(1)5$, $n = 0(1)15$ to 8D.

5. The Sine and Cosine Integrals. For examples of (11)–(13) let

$$\text{(24)} \qquad \begin{matrix} g_1(x) \\ g_2(x) \end{matrix} = \begin{matrix} (1 - \cos ax)/x \\ \sin ax/x \end{matrix} = \int_0^{\infty} f(t) \begin{matrix} \sin xt \\ \cos xt \end{matrix}\, dt,$$

$$\text{(25)} \qquad f(t) = \begin{cases} 1, & 0 < x < a, \\ 0, & a < x < \infty. \end{cases}$$

Using [2, v. 2, p. 333, No. (1)] to evaluate (12) and (13) for $\alpha = -\frac{1}{2}$, we find that

$$\text{(26)} \qquad S_n = \begin{cases} 0, & n \text{ even}, \\ 4e^{(n-1)[\pi i/2]} \displaystyle\sum_{k=0}^{\infty} J_{n+2k+1}(a), & n \text{ odd}, \end{cases}$$

$$\text{(27)} \qquad C_n = \begin{cases} 0, & n \text{ odd}, \\ 2\epsilon_n e^{n\pi i/2} \displaystyle\sum_{k=0}^{\infty} J_{n+2k+1}(a), & n \text{ even}. \end{cases}$$

Let $a = 2$ and 5 in (26) and (27), and use [1, v. 2, p. 145, No. (6)] and (23) to obtain the expansion whose coefficients are listed in Table 2.

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